ITÔ'S FORMULA FOR BANACH SPACE VALUED JUMP PROCESSES DRIVEN BY POISSON RANDOM MEASURES

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ABSTRACT. We prove Itô's formula for a general class of functions $H : \mathbb{R}_+ \times F \to G$ of class $C^{1,2}$, where F, G are separable Banach spaces, and jump processes driven by a compensated Poisson random measure.

1. Introduction

We prove Itô's formula for Banach space valued stochastic jump processes driven by a compensated Poisson random measure. In this context, Itô's formula was originally given in [14]. However, there it was only proven for a smaller class of integrands, as stochastic integration for Banach space valued integrands was still not understood in the generality of the forthcoming papers [9], [10]. We remind here also the work of E. Hausenblas [6], where Itô's formula on Banach spaces was proven assuming additional conditions on the integrands. In a previous work of Graveraux and Pellaumail [5], where also additional conditions on the integrands are required, the Itô formula was not given in terms of the compensator. However, this is necessary in case the formula is used to find the generator of a Markov process. In this article, we provide an improvement of the work of [14]. Even if the methods are similar, the current article is presented in a clearer and direct way, due to integrands having general integrability conditions. The mark space $(E, \mathcal{E})$, associated to the compensated Poisson random measure, is also allowed to be more general than in [14], where E has been a separable Banach space. We refer Remark 2.1 and Remark 3.2, where this generalization is discussed. Moreover, an additional improvement is given, which is important for applications: The condition $H \in C_b^{1,2}(\mathbb{R}_+ \times F; G)$, i.e. that the function H and its partial derivatives are bounded, is not required any more. This means that Itô's formula can be applied to functionals such as $\| \cdot \|^2$, as discussed in Example 3.9.

2. Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, and let $(F, \| \cdot \|)$ be a separable Banach space. Let $(E, \mathcal{E})$ be a measurable space which we assume to be a *Blackwell space* (see [1, 4]). Furthermore, let N be a time-homogeneous Poisson random measure on $\mathbb{R}_+ \times E$, see [7, Def. II.1.20]. Then its compensator is of the form $\nu(dt, dx) = dt\beta(dx)$, where β is a σ-finite measure on $(E, \mathcal{E})$. We call $q(dt, dx) = N(dt, dx) - \nu(dt, dx)$ the associated compensated Poisson random measure.

Remark 2.1. *In previous works on this topic, see e.g.* [13, 14, 9, 10]*, the mark space $(E, \mathcal{E})$ is a separable Banach space equipped with its Borel σ-field, and N is*

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the Poisson random measure associated to an E-valued Lévy process with Lévy measure β. The upcoming results from this Section, which we take from [13, 14, 9, 10], *also hold true in our present, more general framework, and with analogous proofs (see, e.g.* [8, 2] *for the case where F is a separable Hilbert space). Indeed, the crucial point is whether the integral operator (3) can be extended to a continuous linear operator, and this property does not rely on the structure of E (see also Remark 2.4 below). The assumption that the measurable space $(E, \mathcal{E})$ is a Blackwell space is a standard assumption when dealing with random measures, see e.g.* [7, Chapter II.1]. *It ensures the existence and uniqueness of the predictable compensator, see* [7, Thm. II.1.8]. *We remark that every Polish space with its Borel σ-field is a Blackwell space.*

We fix an arbitrary $T \in \mathbb{R}_+$. Let us consider the set of progressively measurable functions on the time interval $[0,T]$, i.e.

$$\begin{aligned} M^T(E/F) := \{ f : \Omega \times [0,T] \times E \to F : f \text{ is } \mathcal{B}[0,T] \otimes \mathcal{E} \otimes \mathcal{F}_T\text{-measurable} \\ \text{and } f(t,x) \text{ is } \mathcal{F}_t\text{-measurable for all } t \in [0,T] \text{ and } x \in E \}. \end{aligned}$$

Furthermore, we define

$$M_\nu^{T,2}(E/F) := \left\{ f \in M^T(E/F) : \int_0^T \int_E \mathbb{E}[\|f(t,x)\|^2] \nu(dt,dx) < \infty \right\},$$

where $\mathbb{E}[f]$ denotes the expectation with respect to the probability measure $\mathbb{P}$.

Definition 2.2. *A function $f \in M^T(E/F)$ belongs to the set $\Sigma_T(E/F)$* of simple functions*, if there exist $n, m \in \mathbb{N}$ such that*

$$f(t,x) = \sum_{k=1}^{n-1} \sum_{l=1}^{m} \mathbb{1}_{A_{k,l}}(x) \mathbb{1}_{F_{k,l}} \mathbb{1}_{(t_k,t_{k+1}]}(t) a_{k,l},$$

where $\beta(A_{k,l}) < \infty$, $t_k \in (0,T]$, $t_k < t_{k+1}$, $F_{k,l} \in \mathcal{F}_{t_k}$, $a_{k,l} \in F$, and for all $k \in 1 \ldots, n-1$ we have $A_{k,l_1} \times F_{k,l_1} \cap A_{k,l_2} \times F_{k,l_2} = \emptyset$ if $l_1 \neq l_2$.

The set $\Sigma_T(E/F)$ of simple functions is dense in the Banach space $M_\nu^{T,2}(E/F)$ with norm

$$\|f\|_2 := \sqrt{\int_0^T \int_E \mathbb{E}[\|f(t,u)\|^2] \nu(dt,dx)}.$$

The proof only uses the fact that the simple functions are defined on the sets of a semiring which generates the σ-algebra $\mathcal{B}[0,T] \otimes \mathcal{E} \otimes \mathcal{F}_T$ and that the compensator is of the form $\nu(dt,dx) = dt\beta(dx)$, see [13, Theorem 4.2] (where this is proven for slightly more general compensated Poisson random measures having compensators $\alpha(dt)\beta(dx)$ with $\alpha(dt)$ being absolutely continuous w.r.t Lebesgue measure). We remark that in [15, Chapter 2.4], which deals the case of real-valued integrands, a bigger set of simple functions is considered.

The Itô integral of simple functions is defined as usual pathwise in a very natural way (see Chapter 3 in [13]): For $f \in \Sigma_T(E/F)$, the "natural stochastic integral" of f is given by

$$\int_0^T \int_E f(t,x) q(dt,dx) := \sum_{k=1}^{n-1} \sum_{l=1}^{m} a_{k,l} \mathbb{1}_{F_{k,l}} q((t_k,t_{k+1}] \cap (0,T] \times A_{k,l}).$$

Remark 2.3. *Suppose the mark space $(E, \mathcal{E})$ is a separable Banach space equipped with its Borel σ-field. Then, for each $f \in \Sigma_T(E/F)$ we have*

$$\int_0^T \int_E f(s,x) q(ds,dx) = \sum_{0<s\leq T} f(s,\Delta X_s) - \mathbb{E}\Bigg[\sum_{0<s\leq T} f(s,\Delta X_s) \Bigg], \tag{1}$$

where $(X_t)_{t\geq 0}$ is the Lévy process associated to the compensated Poisson random measure $q(ds, dx)$, and ΔX_s denotes the jump of X at time s, i.e. $\Delta X_s = X_s - \lim_{u\uparrow s} X_u$.

Let $\mathcal{M}_T^2(F)$ be the linear space of all F-valued square integrable martingales $M = (M_t)_{t\in[0,T]}$ with norm

$$\|M\|_{\mathcal{M}_T^2} = \big(\mathbb{E}[\|M_T\|^2]\big)^{1/2}.$$

The Itô integral for functions $f \in M_\nu^{T,2}(E/F)$ is well defined, if the linear operator

$$\Sigma_T(E/F) \to \mathcal{M}_T^2(F), \quad f \mapsto \left(\int_0^t \int_E f(s,x)q(ds,dx)\right)_{t\in[0,T]} \tag{2}$$

can uniquely be extended to a continuous linear operator

$$M_\nu^{T,2}(E/F) \to \mathcal{M}_T^2(F), \quad f \mapsto \left(\int_0^t \int_E f(s,x)q(ds,dx)\right)_{t\in[0,T]}. \tag{3}$$

In particular, if this is the case, for all $f \in M_\nu^{T,2}(E/F)$ there is a sequence $(f_n)_{n\in\mathbb{N}} \subset \Sigma_T(E/F)$ such that $\lim_{n\to\infty} \|f - f_n\|_2 = 0$ and

$$\lim_{n\to 0} \mathbb{E}\left[\left\| \int_0^T \int_E (f(s,x) - f_n(s,x))q(ds,dx)\right\|^2\right] = 0.$$

Remark 2.4. *In [10] we have proven that the Itô Integral for simple functions in (2) can uniquely be extended to a continuous linear operator (3) if and only if there is a constant K_β, depending on the Lévy measure β, such that*

$$\begin{aligned}&\mathbb{E}\left[\left\|\int_0^T \int_E f(s,x)q(ds,dx)\right\|^2\right] \\ &\leq K_\beta \mathbb{E}\left[\int_0^T \int_E \|f(s,x)\|^2 \beta(dx)ds\right] \quad \textit{for all } f \in \Sigma_T(E/F).\end{aligned} \tag{4}$$

If the Banach space F is of M-type 2, then such a constant exists and does not depend on the Lévy measure β, see [9]. We also emphasize that – as indicated in Remark 2.1 – the proof of this result does not rely on the structure of E. Hence, the M-type 2 condition is a sufficient, but not necessary condition for the Itô integral with respect to compensated Poisson random measures to be well defined. Typical examples of M-type 2 Banach spaces are the Lebesgue spaces $L^p(G, \mathcal{G}, \mu)$ with $2 \leq p < \infty$ and $(G, \mathcal{G}, \mu)$ being a measure space, see [11], [12]. On the other hand, if such a constant exists and does not depend on the Lévy measure β, i.e. $K_\beta = K$, then the Banach space F has to be of type 2, see [10]. We also remark that in case $F = H$ being a Hilbert space, the linear operator (3) is even an isometry, see [13].

There are separable Banach spaces which are not of M-type 2, as the following example shows:

Example 2.5. *Let ℓ^1 be the space of all real-valued sequences $(x_j)_{j\in\mathbb{N}} \subset \mathbb{R}$ which are absolutely convergent, that is*

$$\|x\|_{\ell^1} := \sum_{j=1}^\infty |x_j| < \infty.$$

Then $(\ell^1, \|\cdot\|_{\ell^1})$ is a separable Banach space which is not of M-type 2. Indeed, let $(e_j)_{j\in\mathbb{N}}$ be the standard unit sequences in ℓ^1, which are given by

$$e_1 = (1, 0, \dots), \quad e_2 = (0, 1, 0, \dots), \quad \dots$$

Let $n \in \mathbb{N}$ be arbitrary. We denote by $(X_j^{(n)})_{j=1,\dots,n}$ independent random variables having a normal distribution $N(0,1/n)$, and we define the ℓ^1-valued process $M^{(n)} = (M_k^{(n)})_{k=0,\dots,n}$ as

$$M_0^{(n)} := 0 \quad \textit{and} \quad M_k^{(n)} := \sum_{j=1}^k X_j^{(n)} e_j, \quad k = 1,\dots,n.$$

Then $M^{(n)}$ is a martingale with respect to the filtration $(\mathcal{F}_k^{(n)})_{k=0,\dots,n}$ given by

$$\mathcal{F}_0^{(n)} = \{\emptyset, \Omega\} \quad \textit{and} \quad \mathcal{F}_k^{(n)} = \sigma(X_1^{(n)},\dots,X_k^{(n)}), \quad k = 1,\dots,n.$$

Moreover, we have

$$\sum_{k=1}^n \mathbb{E}\big[\|M_k^{(n)} - M_{k-1}^{(n)}\|_{\ell^1}^2\big] = \sum_{k=1}^n \mathbb{E}[\|X_k^{(n)} e_k\|_{\ell^1}^2] = \sum_{k=1}^n \mathbb{E}\big[|X_k^{(n)}|^2\big] = 1$$

as well as

$$\begin{aligned}
\mathbb{E}\big[\|M_n^{(n)}\|_{\ell^1}^2\big] &= \mathbb{E}\Bigg[\bigg\|\sum_{j=1}^n X_j^{(n)} e_j\bigg\|_{\ell^1}^2\Bigg] = \mathbb{E}\Bigg[\bigg(\sum_{j=1}^n |X_j^{(n)}|\bigg)^2\Bigg] \\
&= \sum_{i=1}^n \sum_{j=1}^n \mathbb{E}\big[|X_i^{(n)} X_j^{(n)}|\big] = \sum_{j=1}^n \mathbb{E}\big[|X_j^{(n)}|^2\big] + \sum_{i \neq j} \mathbb{E}\big[|X_i^{(n)}|\big]\mathbb{E}\big[|X_j^{(n)}|\big] \\
&= 1 + \sum_{i \neq j} \frac{2}{\pi n} = 1 + \frac{2n(n-1)}{\pi n} = 1 + \frac{2(n-1)}{\pi} \to \infty \quad \textit{for } n \to \infty,
\end{aligned}$$

showing that ℓ^1 is not of M-type 2.

Remark 2.6. *The space ℓ^1 has also been utilized in* [17] *in order to provide counterexamples for stochastic integration in Banach spaces with respect to a Wiener process. In* [16]*, the stochastic integral with respect to a Wiener process has been defined on so-called UMD Banach spaces. This approach is based on a two-sided decoupling inequality for UMD spaces due to* [3].

Together with Example 2.5, the next result shows that ℓ^1 is a separable Banach space, which is not of M-type 2, but where inequality (4) is satisfied for certain Poisson random measures.

Proposition 2.7. *We suppose that $\beta(E) < \infty$. Then inequality (4) is satisfied with $K_\beta = 4 + 6T\beta(E)$.*

Proof. Let $f \in \Sigma_T(E/F)$ be arbitrary. Then we have

$$\begin{aligned}
&\mathbb{E}\Bigg[\bigg\|\int_0^T \int_E f(s,x) q(ds,dx)\bigg\|^2\Bigg] \\
&\leq 2\mathbb{E}\Bigg[\bigg(\int_0^T \int_E \|f(s,x)\| N(ds,dx)\bigg)^2\Bigg] + 2\mathbb{E}\Bigg[\bigg(\int_0^T \int_E \|f(s,x)\|\beta(dx)ds\bigg)^2\Bigg] \\
&\leq 2\mathbb{E}\Bigg[\bigg(\int_0^T \int_E \|f(s,x)\| q(ds,dx) + \int_0^T \int_E \|f(s,x)\|\beta(dx)ds\bigg)^2\Bigg] \\
&\quad + 2\mathbb{E}\Bigg[\bigg(\int_0^T \int_E \|f(s,x)\|\beta(dx)ds\bigg)^2\Bigg].
\end{aligned}$$

Thus, by the Itô isometry for real-valued integrands and the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg\|\int_0^T\int_E f(s,x)q(ds,dx)\bigg\|^2\bigg] \\
&\le 4\mathbb{E}\bigg[\bigg(\int_0^T\int_E \|f(s,x)\|q(ds,dx)\bigg)^2\bigg] + 6\mathbb{E}\bigg[\bigg(\int_0^T\int_E \|f(s,x)\|\beta(dx)ds\bigg)^2\bigg] \\
&\le 4\mathbb{E}\bigg[\int_0^T\int_E \|f(s,x)\|^2\beta(dx)ds\bigg] + 6T\beta(E)\mathbb{E}\bigg[\int_0^T\int_E \|f(s,x)\|^2\beta(dx)ds\bigg].
\end{aligned}$$

Consequently, inequality (4) is satisfied with $K_\beta = 4 + 6T\beta(E)$. □

If the continuous linear operator (3) is well defined, then the definition of the Itô integral can be extended to all $f \in \mathcal{K}^2_{T,\beta}(E/F)$, where $\mathcal{K}^2_{T,\beta}(E/F)$ denotes the linear space of all progressively measurable $f \in M^T(E/F)$ such that

$$\mathbb{P}\bigg(\int_0^T\int_E \|f(s,x)\|^2\beta(dx)ds < \infty\bigg) = 1.$$

For all $f \in \mathcal{K}^2_{T,\beta}(F)$ we define the sequence of stopping times

$$\tau_n := \inf\bigg\{t \in [0,T] : \int_0^t\int_E \|f(s,x)\|^2\beta(dx)ds \ge n\bigg\}, \quad n \in \mathbb{N}.$$

Note that $f\mathbb{1}_{[0,\tau_n]} \in M^{T,2}_\nu(E/F)$ for all $n \in \mathbb{N}$. Hence, we can define the Itô integral

$$\int_0^t\int_E f(s,x)q(ds,dx) := \lim_{n\to\infty}\int_0^t\int_E f(s,x)\mathbb{1}_{[0,\tau_n]}q(ds,dx), \quad t \in [0,T]$$

which is a local martingale.

In the sequel we will use Theorem 7.7 with Remark 7.8 from [13]. We recall the result here:

Theorem 2.8. *Let $f \in \mathcal{K}^2_{T,\beta}(E/F)$ be arbitrary and let $(f_n)_{n\in\mathbb{N}}$ be a sequence such that $f_n \in \mathcal{K}^2_{T,\beta}(E/F)$ for all $n \in \mathbb{N}$. Suppose that f_n converges $\nu\otimes\mathbb{P}$–almost surely to f on $\Omega\times[0,T]\times E$, when $n\to\infty$, and $\mathbb{P}$–almost surely*

$$\lim_{n\to\infty}\int_0^T\int_E \|f_n - f\|^2 d\nu = 0.$$

Assume there is $g \in \mathcal{K}^2_{T,\beta}(E/F)$ such that

$$\int_0^T\int_E \|f_n\|^2 d\nu \le \int_0^T\int_E \|g\|^2 d\nu.$$

Then, we have

$$\int_0^t\int_E f(s,x)q(ds,dx) = \lim_{n\to\infty}\int_0^t\int_E f_n(s,x)q(ds,dx),$$

where the limit is in probability.

3. Itô's formula for Banach space valued functions

Let F be a separable Banach space and the integral operator (3) be a continuous linear operator for each $T \in \mathbb{R}_+$. By Remark 2.4 this is equivalent to state that there is a constant K_β such that (4) holds. As pointed out in Remark 2.4, this is in particular satisfied when the Banach space F is of M-type 2.

Remark 3.1. *According to* [7, Prop. II.1.14]*, there exist a sequence* $(\tau_k)_{k\in\mathbb{N}}$ *of finite stopping times with* $[\![\tau_k]\!] \cap [\![\tau_l]\!] = \emptyset$ *for* $k \neq l$ *and an* E*-valued optional process* ξ *such that for every optional process* $f : \Omega \times \mathbb{R}_+ \times E \to H$ *with*

$$\mathbb{P}\bigg(\int_0^t \int_E \|f(s,x)\| N(ds,dx) < \infty \bigg) = 1 \quad \textit{for all } t \geq 0$$

we have the identity

$$\int_0^t \int_E f(s,x) N(ds,dx) = \sum_{k\in\mathbb{N}} f(\tau_k, \xi_{\tau_k}) \mathbb{1}_{\{\tau_k \leq t\}}, \quad t \geq 0. \tag{5}$$

Remark 3.2. *Suppose the mark space* $(E, \mathcal{E})$ *is a separable Banach space equipped with its Borel* σ*-field. From Remark 2.3 it follows that the stopping times* τ_k *in Remark 3.1 can be chosen to be the jump times of the corresponding Lévy process* $(X_t)_{t\geq 0}$*, with the random variables* ξ_{τ_k} *being the jumps of the process at time* τ_k*, that is* $\xi = \Delta X$*. An analogous statement for* $E = \mathbb{R}^d$ *and an adapted càdlàg process* X *can be found in* [7, Prop.II.1.16]*. The corresponding result, where* E *is a separable Banach space, is given by Theorem 5.1 in* [13]*. The result* [7, Prop. II.1.14] *used in Remark 3.1 allows us to use a more general mark space* $(E, \mathcal{E})$ *than in* [13]*, i.e. a Blackwell space.*

From now on, let G be another separable Banach space such that integral operator (3) with $F = G$ is a continuous linear operator for each $T \in \mathbb{R}_+$. Again by Remark 2.4, this ensures that all upcoming stochastic integrals are well defined, and that, for some constant $K_\beta > 0$, for each $T \in \mathbb{R}_+$ we have the estimates

$$\mathbb{E}\bigg[\bigg\| \int_0^T \int_E f(s,x) q(ds,dx) \bigg\|^2 \bigg] \leq K_\beta E\bigg[\int_0^T \int_E \|f(s,x)\|^2 ds \beta(dx) \bigg]$$

for all $f \in M_\nu^{T,2}(E/F)$ and all $f \in M_\nu^{T,2}(E/G)$. We start with a version of Itô's formula, where the mark space is finite. Based on this result, we shall prove Theorem 3.6 later on.

Proposition 3.3. *We suppose that:*

- $H \in C^{1,2}(\mathbb{R}_+ \times F; G)$ *is a function.*
- $C \in \mathcal{E}$ *is a set with* $\beta(C) < \infty$.
- $f : \Omega \times \mathbb{R}_+ \times E \to F$ *is a progressively measurable process.*
- $g : \Omega \times \mathbb{R}_+ \times E \to F$ *is a progressively measurable process such that for all* $t \in \mathbb{R}_+$ *we have* $\mathbb{P}$*–almost surely*

$$\int_0^t \int_C \|g(s,x)\| \nu(ds,dx) < \infty. \tag{6}$$

- Y *is an Itô process of the form*

$$Y_t = Y_0 + \int_0^t \int_C f(s,x) N(ds,dx) + \int_0^t \int_C g(s,x) \nu(ds,dx), \quad t \geq 0.$$

Then, the following statements are true:

(1) *For all* $t \in \mathbb{R}_+$ *we have* $\mathbb{P}$*–almost surely*

$$\int_0^t \|\partial_s H(s, Y_s)\| ds < \infty, \tag{7}$$

$$\int_0^t \int_C \|H(s, Y_{s-} + f(s,x)) - H(s, Y_{s-})\| N(ds,dx) < \infty, \tag{8}$$

$$\int_0^t \int_C \|\partial_y H(s, Y_s) g(s,x)\| \nu(ds,dx) < \infty. \tag{9}$$

(2) *We have $\mathbb{P}$–almost surely*

$$\begin{aligned} H(t, Y_t) = H(0, Y_0) &+ \int_0^t \partial_s H(s, Y_s) ds \\ &+ \int_0^t \int_C \big(H(s, Y_{s-} + f(s,x)) - H(s, Y_{s-}) \big) N(ds, dx) \\ &+ \int_0^t \int_C \partial_y H(s, Y_s) g(s,x) \nu(ds, dx), \quad t \geq 0. \end{aligned}$$

Proof. Estimates (7), (9) hold true by (6) and the continuity of the partial derivatives $\partial_s H$, $\partial_y H$, and (8) is valid, because $\beta(C) < \infty$. We define the Itô processes

$$\begin{aligned} Y_t^N &:= Y_0 + \int_0^t \int_C f(s,x) N(ds, dx), \quad t \geq 0 \\ Z_t^\nu &:= \int_0^t \int_C g(s,x) \nu(ds, dx), \quad t \geq 0. \end{aligned}$$

Let $h \in C^{1,2,2}(\mathbb{R}_+ \times F \times F; G)$ be the function $h(t,y,z) := H(t, y+z)$. Furthermore, let $(\Pi_n)_{n \in \mathbb{N}}$ be a sequence of decompositions of $\mathbb{R}_+$ with $|\Pi_n| \to 0$. Let $t \in \mathbb{R}_+$ be arbitrary. Then, we have

$$\begin{aligned} &H(t, Y_t) - H(0, Y_0) = h(t, Y_t^N, Z_t^\nu) - h(0, Y_0^N, Z_0^\nu) \\ &= \lim_{n\to\infty} \sum_{t_i \in \Pi_n} \big(h(t_{i+1} \wedge t, Y_{t_{i+1}\wedge t}^N, Z_{t_{i+1}\wedge t}^\nu) - h(t_i, Y_{t_i}^N, Z_{t_i}^\nu) \big) \\ &= \lim_{n\to\infty} \sum_{t_i \in \Pi_n} \big(h(t_{i+1} \wedge t, Y_{t_{i+1}\wedge t}^N, Z_{t_{i+1}\wedge t}^\nu) - h(t_i, Y_{t_{i+1}\wedge t}^N, Z_{t_{i+1}\wedge t}^\nu) \big) \\ &\quad + \lim_{n\to\infty} \sum_{t_i \in \Pi_n} \big(h(t_i, Y_{t_{i+1}\wedge t}^N, Z_{t_{i+1}\wedge t}^\nu) - h(t_i, Y_{t_i}^N, Z_{t_{i+1}\wedge t}^\nu) \big) \\ &\quad + \lim_{n\to\infty} \sum_{t_i \in \Pi_n} \big(h(t_i, Y_{t_i}^N, Z_{t_{i+1}\wedge t}^\nu) - h(t_i, Y_{t_i}^N, Z_{t_i}^\nu) \big). \end{aligned}$$

By Taylor's theorem, the first term equals

$$\begin{aligned} &\lim_{n\to\infty} \sum_{t_i \in \Pi_n} \big(h(t_{i+1} \wedge t, Y_{t_{i+1}\wedge t}^N, Z_{t_{i+1}\wedge t}^\nu) - h(t_i, Y_{t_{i+1}\wedge t}^N, Z_{t_{i+1}\wedge t}^\nu) \big) \\ &= \lim_{n\to\infty} \sum_{t_i \in \Pi_n} \int_0^1 \partial_s h(t_i + \theta((t_{i+1}\wedge t) - t_i), Y_{t_{i+1}\wedge t}^N, Z_{t_{i+1}\wedge t}^\nu)((t_{i+1}\wedge t) - t_i) d\theta \\ &= \lim_{n\to\infty} \int_0^t \bigg(\sum_{t_i \in \Pi_n} \int_0^1 \partial_s h(t_i + \theta((t_{i+1}\wedge t) - t_i), Y_{t_{i+1}\wedge t}^N, Z_{t_{i+1}\wedge t}^\nu) d\theta \\ &\qquad \mathbb{1}_{(t_i, t_{i+1}\wedge t]}(s) \bigg) ds. \end{aligned}$$

Using Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned} &\lim_{n\to\infty} \sum_{t_i \in \Pi_n} \big(h(t_{i+1} \wedge t, Y_{t_{i+1}\wedge t}^N, Z_{t_{i+1}\wedge t}^\nu) - h(t_i, Y_{t_{i+1}\wedge t}^N, Z_{t_{i+1}\wedge t}^\nu) \big) \\ &= \int_0^t \partial_s h(s, Y_s^N, Z_s^\nu) ds = \int_0^t \partial_s H(s, Y_s) ds. \end{aligned}$$

By Remark 3.1 we have

$$Y_t^N = Y_0 + \sum_{k \in \mathbb{N}} f(\tau_k, \xi_{\tau_k}) \mathbb{1}_{\{\tau_k \leq t\}}, \quad t \geq 0.$$

Therefore, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{t_i \in \Pi_n} \big(h(t_i, Y^N_{t_{i+1}\wedge t}, Z^\nu_{t_{i+1}\wedge t}) - h(t_i, Y^N_{t_i}, Z^\nu_{t_{i+1}\wedge t}) \big) \\
&= \lim_{n\to\infty} \sum_{t_i \in \Pi_n} \sum_{k\in\mathbb{N}} \big(h(t_i, Y^N_{\tau_k}, Z^\nu_{t_{i+1}\wedge t}) - h(t_i, Y^N_{\tau_k -}, Z^\nu_{t_{i+1}\wedge t}) \big) \mathbb{1}_{(t_i, t_{i+1}\wedge t]}(\tau_k) \\
&= \lim_{n\to\infty} \sum_{t_i \in \Pi_n} \sum_{k\in\mathbb{N}} \big(h(t_i, Y^N_{\tau_k -} + f(\tau_k, \xi_{\tau_k}), Z^\nu_{t_{i+1}\wedge t}) - h(t_i, Y^N_{\tau_k -}, Z^\nu_{t_{i+1}\wedge t}) \big) \\
&\qquad \mathbb{1}_{(t_i, t_{i+1}\wedge t]}(\tau_k) \\
&= \lim_{n\to\infty} \sum_{t_i \in \Pi_n} \int_{t_i}^{t_{i+1}\wedge t} \int_C \big(h(t_i, Y^N_{s-} + f(s,x), Z^\nu_{t_{i+1}\wedge t}) - h(t_i, Y^N_{s-}, Z^\nu_{t_{i+1}\wedge t}) \big) \\
&\qquad N(ds,dx).
\end{aligned}$$

Consequently, by Lebesgue's dominated convergence theorem, the second term equals

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{t_i \in \Pi_n} \big(h(t_i, Y^N_{t_{i+1}\wedge t}, Z^\nu_{t_{i+1}\wedge t}) - h(t_i, Y^N_{t_i}, Z^\nu_{t_{i+1}\wedge t}) \big) \\
&= \lim_{n\to\infty} \int_0^t \int_C \bigg(\sum_{t_i \in \Pi_n} \big(h(t_i, Y^N_{s-} + f(s,x), Z^\nu_{t_{i+1}\wedge t}) - h(t_i, Y^N_{s-}, Z^\nu_{t_{i+1}\wedge t}) \big) \\
&\qquad \mathbb{1}_{(t_i, t_{i+1}\wedge t]}(s) \bigg) N(ds,dx) \\
&= \int_0^t \int_C \big(h(s, Y^N_{s-} + f(s,x), Z^\nu_s) - h(s, Y^N_{s-}, Z^\nu_s) \big) N(ds,dx) \\
&= \int_0^t \int_C \big(H(s, Y_{s-} + f(s,x)) - H(s, Y_{s-}) \big) N(ds,dx).
\end{aligned}$$

By Taylor's theorem, for the third term we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{t_i \in \Pi_n} \big(h(t_i, Y^N_{t_i}, Z^\nu_{t_{i+1}\wedge t}) - h(t_i, Y^N_{t_i}, Z^\nu_{t_i}) \big) \\
&= \lim_{n\to\infty} \sum_{t_i \in \Pi_n} \int_0^1 \partial_z h(t_i, Y^N_{t_i}, Z^\nu_{t_i} + \theta(Z^\nu_{t_{i+1}\wedge t} - Z^\nu_{t_i}))(Z^\nu_{t_{i+1}\wedge t} - Z^\nu_{t_i}) d\theta \\
&= \lim_{n\to\infty} \sum_{t_i \in \Pi_n} \int_0^1 \partial_z h(t_i, Y^N_{t_i}, Z^\nu_{t_i} + \theta(Z^\nu_{t_{i+1}\wedge t} - Z^\nu_{t_i})) \int_{t_i}^{t_{i+1}\wedge t} \int_C g(s,x) \nu(ds,dx) d\theta \\
&= \lim_{n\to\infty} \sum_{t_i \in \Pi_n} \int_{t_i}^{t_{i+1}\wedge t} \int_C \int_0^1 \partial_z h(t_i, Y^N_{t_i}, Z^\nu_{t_i} + \theta(Z^\nu_{t_{i+1}\wedge t} - Z^\nu_{t_i})) g(s,x) d\theta \nu(ds,dx).
\end{aligned}$$

Therefore, by Lebesgue's dominated convergence theorem we get

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{t_i \in \Pi_n} \big(h(t_i, Y^N_{t_i}, Z^\nu_{t_{i+1}\wedge t}) - h(t_i, Y^N_{t_i}, Z^\nu_{t_i}) \big) \\
&= \lim_{n\to\infty} \int_0^t \int_C \int_0^1 \bigg(\sum_{t_i \in \Pi_n} \partial_z h(t_i, Y^N_{t_i}, Z^\nu_{t_i} + \theta(Z^\nu_{t_{i+1}\wedge t} - Z^\nu_{t_i})) g(s,x) d\theta \\
&\qquad \mathbb{1}_{(t_i, t_{i+1}\wedge t]}(s) \bigg) \nu(ds,dx) \\
&= \int_0^t \int_C \partial_z h(s, Y^N_s, Z^\nu_s) g(s,x) \nu(ds,dx) = \int_0^t \int_C \partial_y H(s, Y_s) g(s,x) \nu(ds,dx).
\end{aligned}$$

This completes the proof. □

Remark 3.4. *In the proof of Proposition 3.3 we used the representation (5), whereas in* [14] *we have used the natural representation (1) of simple functions. Due to Remark 3.2, these methods are more or less equivalent, but the proof here is shorter and allows a more general mark space* $(E, \mathcal{E})$.

Definition 3.5. *We call a continuous, non-decreasing function* $h : \mathbb{R}_+ \to \mathbb{R}_+$ quasi-sublinear *if there is a constant* $C > 0$ *such that*

$$h(x+y) \leq C\big(h(x) + h(y)\big), \quad x, y \in \mathbb{R}_+,$$
$$h(xy) \leq C h(x) h(y), \quad x, y \in \mathbb{R}_+.$$

Theorem 3.6. *We suppose that:*

- $H \in C^{1,2}(\mathbb{R}_+ \times F; G)$ *is a function such that*

(10) $$\|\partial_y H(s,y)\| \leq h_1(\|y\|), \quad (s,y) \in \mathbb{R}_+ \times F$$

(11) $$\|\partial_{yy} H(s,y)\| \leq h_2(\|y\|), \quad (s,y) \in \mathbb{R}_+ \times F$$

  for quasi-sublinear functions $h_1, h_2 : \mathbb{R}_+ \to \mathbb{R}_+$.
- $B \in \mathcal{E}$ *is a set with* $\beta(B^c) < \infty$.
- $f : \Omega \times \mathbb{R}_+ \times E \to F$ *is a progressively measurable process such that for all* $t \in \mathbb{R}_+$ *we have* $\mathbb{P}$*–almost surely*

(12) $$\begin{aligned} &\int_0^t \int_B \|f(s,x)\|^2 \nu(ds,dx) + \int_0^t \int_B h_1(\|f(s,x)\|)^2 \|f(s,x)\|^2 \nu(ds,dx) \\ &\quad + \int_0^t \int_B h_2(\|f(s,x)\|) \|f(s,x)\|^2 \nu(ds,dx) < \infty. \end{aligned}$$

- $g : \Omega \times \mathbb{R}_+ \times E \to F$ *is a progressively measurable process.*
- Y *is an Itô process of the form*

$$Y_t = Y_0 + \int_0^t \int_B f(s,x) q(ds,dx) + \int_0^t \int_{B^c} g(s,x) N(ds,dx), \quad t \geq 0.$$

Then, the following statements are true:

(1) *For all* $t \in \mathbb{R}_+$ *we have* $\mathbb{P}$*–almost surely*

(13) $$\int_0^t \|\partial_s H(s,Y_s)\| ds < \infty,$$

(14) $$\int_0^t \int_B \|H(s, Y_s + f(s,x)) - H(s,Y_s)\|^2 \nu(ds,dx) < \infty,$$

(15) $$\int_0^t \int_B \|H(s, Y_s + f(s,x)) - H(s,Y_s) - \partial_y H(s,Y_s) f(s,x)\| \nu(ds,dx) < \infty,$$

(16) $$\int_0^t \int_{B^c} \|H(s, Y_{s-} + g(s,x)) - H(s,Y_{s\_})\| N(ds,dx) < \infty.$$

(2) *We have* $\mathbb{P}$*–almost surely*

(17) $$\begin{aligned} H(t,Y_t) &= H(0,Y_0) + \int_0^t \partial_s H(s,Y_s) ds \\ &+ \int_0^t \int_B \big(H(s, Y_{s-} + f(s,x)) - H(s,Y_{s-})\big) q(ds,dx) \\ &+ \int_0^t \int_B \big(H(s, Y_s + f(s,x)) - H(s,Y_s) - \partial_y H(s,Y_s) f(s,x)\big) \nu(ds,dx) \\ &+ \int_0^t \int_{B^c} \big(H(s, Y_{s-} + g(s,x)) - H(s,Y_{s-})\big) N(ds,dx), \quad t \geq 0. \end{aligned}$$

Proof. Estimate (13) holds true by the continuity of the partial derivative $\partial_s H$, and (16) is valid, because $\beta(B^c) < \infty$. By Taylor's theorem, the Cauchy Schwarz inequality and (10), we obtain $\mathbb{P}$–almost surely

$$\begin{aligned}
&\int_0^t \int_B \| H(s, Y_s + f(s,x)) - H(s, Y_s) \|^2 \nu(ds,dx) \\
&= \int_0^t \int_B \left\| \int_0^1 \partial_y H(s, Y_s + \theta f(s,x)) f(s,x) d\theta \right\|^2 \nu(ds,dx) \\
&\leq \int_0^t \int_B \int_0^1 \| \partial_y H(s, Y_s + \theta f(s,x)) \|^2 \| f(s,x) \|^2 d\theta \nu(ds,dx) \\
&\leq \int_0^t \int_B \int_0^1 h_1(\| Y_s + \theta f(s,x)) \|)^2 \| f(s,x) \|^2 d\theta \nu(ds,dx)
\end{aligned}$$

Since h_1 is quasi-sublinear, for some constant $C > 0$ we get $\mathbb{P}$–almost surely

$$\begin{aligned}
&\int_0^t \int_B \| H(s, Y_s + f(s,x)) - H(s, Y_s) \|^2 \nu(ds,dx) \\
&\leq C^2 \int_0^t \int_B \int_0^1 \big(h_1(\|Y_s\|) + C h_1(\theta) h_1(\|f(s,x)\|) \big)^2 \| f(s,x) \|^2 d\theta \nu(ds,dx) \\
&\leq 2C^2 \int_0^t \int_B h_1(\|Y_s\|)^2 \| f(s,x) \|^2 \nu(ds,dx) \\
&\quad + 2C^4 h_1(1) \int_0^t \int_B h_1(\|f(s,x)\|)^2 \| f(s,x) \|^2 \nu(ds,dx) < \infty,
\end{aligned}$$

showing (14). By Taylor's theorem and (11), we obtain $\mathbb{P}$–almost surely

$$\begin{aligned}
&\int_0^t \int_B \| H(s, Y_s + f(s,x)) - H(s, Y_s) - \partial_y H(s, Y_s) f(s,x) \| \nu(ds,dx) \\
&= \int_0^t \int_B \left\| \int_0^1 \partial_{yy} H(s, Y_s + \theta f(s,x))(f(s,x), f(s,x)) d\theta \right\| \nu(ds,dx) \\
&\leq \int_0^t \int_B \int_0^1 \| \partial_{yy} H(s, Y_s + \theta f(s,x)) \| \, \| f(s,x) \|^2 d\theta \nu(ds,dx) \\
&\leq \int_0^t \int_B \int_0^1 h_2(\| Y_s + \theta f(s,x) \|) \| f(s,x) \|^2 d\theta \nu(ds,dx)
\end{aligned}$$

Since h_2 is quasi-sublinear, for some constant $C > 0$ we get $\mathbb{P}$–almost surely

$$\begin{aligned}
&\int_0^t \int_B \| H(s, Y_s + f(s,x)) - H(s, Y_s) - \partial_y H(s, Y_s) f(s,x) \| \nu(ds,dx) \\
&\leq C \int_0^t \int_B \int_0^1 \big(h_2(\|Y_s\|) + C h_2(\theta) h_2(\|f(s,x)\|) \big) \| f(s,x) \|^2 d\theta \nu(ds,dx) \\
&\leq C \int_0^t \int_B h_2(\|Y_s\|) \| f(s,x) \|^2 \nu(ds,dx) \\
&\quad + C^2 h_2(1) \int_0^t \int_B h_2(\|f(s,x)\|) \| f(s,x) \|^2 \nu(ds,dx) < \infty,
\end{aligned}$$

providing (15). Since the measure β is σ-finite, there exists a sequence $(C_n)_{n \in \mathbb{N}} \subset \mathcal{E}$ such that $C^n \uparrow E$ and $\beta(C_n) < \infty$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ let Y^n be the Itô process

$$Y_t^n := Y_0 + \int_0^t \int_{B \cap C_n} f(s,x) q(ds,dx) + \int_0^t \int_{B^c \cap C_n} g(s,x) N(ds,dx), \quad t \geq 0.$$

Then, we can express Y^n as

$$\begin{aligned} Y_t^n = Y_0 &+ \int_0^t \int_{C_n} \big(f(s,x)\mathbb{1}_B(x) + g(s,x)\mathbb{1}_{B^c}(x) \big) N(ds,dx) \\ &- \int_0^t \int_{B \cap C_n} f(s,x) \nu(ds,dx), \quad t \geq 0. \end{aligned}$$

Note that, by the Cauchy-Schwarz inequality and (12), for each $t \in \mathbb{R}_+$ we have

$$\begin{aligned} &\int_0^t \int_{B \cap C_n} \| f(s,x) \| \nu(ds,dx) \\ &\leq \big(t\beta(B \cap C_n) \big)^{1/2} \bigg(\int_0^t \int_{B \cap C_n} \| f(s,x) \|^2 \nu(ds,dx) \bigg)^{1/2} < \infty, \end{aligned}$$

showing that condition (6) with $g = -f\mathbb{1}_B$ and $C = C_n$ is satisfied. Using Proposition 3.3, we obtain $\mathbb{P}$–almost surely

$$\begin{aligned} H(Y_t^n) &= H(Y_0) + \int_0^t \partial_s H(s, Y_s^n) ds \\ &+ \int_0^t \int_{C_n} \big(H(s, Y_{s-}^n + f(s,x)\mathbb{1}_B(x) + g(s,x)\mathbb{1}_{B^c}(x)) - H(s, Y_{s-}^n) \big) N(ds,dx) \\ &- \int_0^t \int_{B \cap C_n} \partial_y H(s, Y_s^n) f(s,x) \nu(ds,dx), \quad t \geq 0. \end{aligned}$$

We can rewrite this formula as

$$\begin{aligned} H(Y_t^n) = H(Y_0) &+ \int_0^t \partial_s H(s, Y_s^n) ds \\ &+ \int_0^t \int_{B \cap C_n} \big(H(s, Y_{s-}^n + f(s,x)) - H(s, Y_{s-}^n) \big) N(ds,dx) \\ &+ \int_0^t \int_{B^c \cap C_n} \big(H(s, Y_{s-}^n + g(s,x)) - H(s, Y_{s-}^n) \big) N(ds,dx) \\ &- \int_0^t \int_{B \cap C_n} \partial_y H(s, Y_s^n) f(s,x) \nu(ds,dx), \quad t \geq 0, \end{aligned}$$

an therefore, we obtain

$$\begin{aligned} H(Y_t^n) &= H(Y_0) + \int_0^t \partial_s H(s, Y_s^n) ds \\ &+ \int_0^t \int_{B \cap C_n} \big(H(s, Y_{s-}^n + f(s,x)) - H(s, Y_{s-}^n) \big) q(ds,dx) \\ &+ \int_0^t \int_{B \cap C_n} \big(H(s, Y_s^n + f(s,x)) - H(s, Y_s^n) - \partial_y H(s, Y_s^n) f(s,x) \big) \nu(ds,dx) \\ &+ \int_0^t \int_{B^c \cap C_n} \big(H(s, Y_{s-}^n + g(s,x)) - H(s, Y_{s-}^n) \big) N(ds,dx), \quad t \geq 0 \end{aligned}$$

Letting $n \to \infty$, by virtue of Theorem 2.8 we arrive at (17). □

Example 3.7. *Suppose that $H \in C_b^{1,2}(\mathbb{R}_+ \times F; G)$ and*

$$\int_0^t \int_B \| f(s,x) \|^2 \nu(ds,dx) < \infty \quad \text{for all } t \in \mathbb{R}_+.$$

Then Theorem 3.6 applies and yields the Itô formula (17), cf. [14].

Example 3.8. *If $H \in L(F,G)$ is a continuous linear operator and*

$$\int_0^t \int_B \|f(s,x)\|^2 \nu(ds,dx) < \infty \quad \textit{for all } t \in \mathbb{R}_+,$$

then Theorem 3.6 applies and yields that $\mathbb{P}$–almost surely

$$\begin{aligned} H(Y_t) = H(Y_0) &+ \int_0^t \int_B H(f(s,x)) q(ds,dx) \\ &+ \int_0^t \int_{B^c} H(g(s,x)) N(ds,dx), \quad t \geq 0. \end{aligned}$$

Example 3.9. *Suppose that F is a separable Hilbert space. Then $H(x) = \|x\|^2$ is of class $C^2(F;\mathbb{R})$ with*

$$H_x(x)v = 2\langle x, v \rangle \quad \textit{and} \quad H_{xx}(x)(v,w) = 2\langle v, w \rangle.$$

Therefore, we have

$$\|H_x(x)\| \leq 2\|x\| \quad \textit{and} \quad \|H_{xx}(x)\| \leq 2.$$

Consequently, if

$$\int_0^t \int_B \|f(s,x)\|^2 \nu(ds,dx) + \int_0^t \int_B \|f(s,x)\|^4 \nu(ds,dx) < \infty \quad \textit{for all } t \in \mathbb{R}_+,$$

then Theorem 3.6 applies and yields that $\mathbb{P}$–almost surely

$$\begin{aligned} \|Y_t\|^2 = \|Y_0\|^2 &+ \int_0^t \int_B \big(\|Y_{s-} + f(s,x)\|^2 - \|Y_{s-}\|^2 \big) q(ds,dx) \\ &+ \int_0^t \int_B \big(\|Y_s + f(s,x)\|^2 - \|Y_s\|^2 - 2\langle Y_s, f(s,x) \rangle \big) \nu(ds,dx) \\ &+ \int_0^t \int_{B^c} \big(\|Y_{s-} + g(s,x)\|^2 - \|Y_{s-}\|^2 \big) N(ds,dx), \quad t \geq 0. \end{aligned}$$

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