Dilatation monotonicity and convex order

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October 2013

Abstract

We study, for functions and sets, the relation between law invariance, preserving the convex or uniform order, and dilatation monotonicity based on duality arguments.

Keywords: law invariance, convex order, uniform order, second order stochastic dominance, dilatation monotonicity
MSC 2010 Classifications: 46N10, 46N30, 91G99.

1 Introduction

The purpose of this note is to exploit a characterization, given in Theorem 2.1, of law invariant convex functions in terms of preservation of the convex order, dilatation monotonicity, and preservation of the uniform preference order. It combines and extends results from Cherny and Grigoriev [2] and Dana [3]. Based on this result we will study law invariant and order monotone sets and functions on $L^p := L^p(\Omega, \mathcal{F}, \mathbb{P})$, $p \in [1, \infty]$, where the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is assumed to be standard and have no atoms. Our main result is an extension of an important result by Cherny and Grigoriev in [2] which states that dilatation monotone, norm continuous, real-valued functions on L^∞ are automatically law invariant. We show that for a lower semicontinuous function on L^p dilatation monotonicity is equivalent to order monotonicity with respect to the convex order (Theorem 2.7). Both conditions imply law invariance of that function. Typical examples of the class of functions we study are law invariant risk measures and law invariant robust utilities. The latter class are also referred to as probabilistic sophisticated variational preferences in the decision sciences. Along the path we also give alternative proofs of results by Ryff [6], Dana [3], and Ekeland and Schachermayer [5] on the convex and uniform order; see Theorem 2.3.

2 Main results

In the following we will write $X \overset{d}{=} Y$ to indicate that the random variables X and Y on the standard atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$ are equally distributed under $\mathbb{P}$, i.e. $\mathrm{law}(X) = \mathrm{law}(Y)$, where $\mathrm{law}(X)$ denotes the distribution of a random variable X under $\mathbb{P}$:

$$\mathrm{law}(X)(A) = \mathbb{P}(X \in A) \quad \text{for all } A \in \mathcal{B}(\mathbb{R}).$$

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Throughout this paper all equalities and inequalities between random variables are understood in the $\mathbb{P}$-almost sure (a.s.) sense. The convex (partial-) order $\succeq$ defined on L^1 (and thus on $L^p \subset L^1$) is given by

$$(X \succeq Y) \quad \Leftrightarrow \quad (E[c(X)] \geq E[c(Y)] \text{ for every convex function } c : \mathbb{R} \to \mathbb{R}).$$

Another important (partial-) order on L^1 is the uniform preference order $\succeq_{uni}$ which is given by

$$(X \succeq_{uni} Y) \quad \Leftrightarrow \quad (E[u(X)] \geq E[u(Y)] \text{ for every increasing concave function } u : \mathbb{R} \to \mathbb{R}).$$

This ordering is also called second order stochastic dominance and plays an important role in economics because $X \succeq_{uni} Y$ means that every expected utility agent prefers the random position X to Y. Thus the preference of X to Y is uniform in the market. In order to be more consistent with the convex order we will however use

$$(X \succeq_{dc} Y) \quad \Leftrightarrow \quad (E[c(X)] \geq E[c(Y)] \text{ for all decreasing convex functions } c : \mathbb{R} \to \mathbb{R})$$

instead of $\succeq_{uni}$. Clearly

$$(X \succeq Y) \quad \Rightarrow \quad (X \succeq_{dc} Y) \quad \Leftrightarrow \quad (X \preceq_{uni} Y). \tag{2.1}$$

For a set $C \subset L^p$ the (convex analytic) indicator function $\delta(\cdot \mid C)$ is defined by

$$\delta(X \mid C) = \begin{cases} 0 & \text{if } X \in C \\ \infty & \text{otherwise} \end{cases}.$$

We call a function $f : L^p \to [-\infty, \infty]$ lower semicontinuous (lsc) if its level sets $E_k := \{X \in L^p \mid f(X) \leq k\}$, $k \in \mathbb{R}$, are all closed in the norm topology induced by the p-norm $\|\cdot\|_p = E[|\cdot|^p]^{1/p}$ on L^p. This is equivalent to $f(X) \leq \liminf_{n\to\infty} f(X_n)$ whenever the sequence $(X_n)_{n\in\mathbb{N}} \subset L^p$ converges to X in $(L^p, \|\cdot\|_p)$. A main ingredient to our studies is the following characterization of law invariance. It basically combines results from [2] and [3].

Theorem 2.1. *Let $f : L^p \to (-\infty, \infty]$ be a convex, lsc function where $p \in [1, \infty]$. Then the following are equivalent:*

- *(i) f is law invariant, i.e. $f(X) = f(Y)$ whenever $X \overset{d}{=} Y$.*
- *(ii) f is $\succeq$-monotone, i.e. $X \succeq Y$ implies $f(X) \geq f(Y)$.*
- *(iii) f is dilatation monotone, i.e. $f(E[X \mid \mathcal{G}]) \leq f(X)$ for all $X \in L^p$ and all sub-σ-algebras $\mathcal{G} \subset \mathcal{F}$.*

If f is in addition antitone with respect to the $\mathbb{P}$-a.s. order, i.e. $X \geq Y$ implies that $f(X) \leq f(Y)$, then either of the conditions (i), (ii) or (iii) above is equivalent to

- *(iv) f is $\succeq_{dc}$-monotone, i.e. $X \succeq_{dc} Y$ implies $f(X) \geq f(Y)$.*

Proof of Theorem 2.1. The relations (i) $\Leftrightarrow$ (ii) and, in case f is antitone with respect to the $\mathbb{P}$-a.s. order, (i) $\Leftrightarrow$ (iv) are proved in [3, Theorem 4.1].

(ii) $\Rightarrow$ (iii): follows from Jensen's inequality.

(iii) $\Rightarrow$ (i): First, it follows from results in [2] that $f|_{L^\infty}$ is law invariant. Indeed Cherny and Grigoriev show in [2] that for every $X, Y \in L^\infty$ with $X \overset{d}{=} Y$ and $\epsilon > 0$ there exists a finite sequence $(\mathcal{G}_k)_{k=1}^K$ of sub-σ-algebras of $\mathcal{F}$ such that

$$\|X - E[\ldots E[E[Y \mid \mathcal{G}_1] \mid \mathcal{G}_2] \ldots \mid \mathcal{G}_K]\|_\infty < \epsilon.$$

Therefore, we may choose a sequence $(\mathcal{G}_k^n)_{k=1}^{K(n)}$ of such finite sequences of sub-σ-algebras such that the sequence

$$Y_n := E[\ldots E[Y \mid \mathcal{G}_1^n] \ldots \mid \mathcal{G}_{K(n)}^n],\ n \in \mathbb{N},$$

converges to X. As $\|\cdot\|_\infty$-convergence implies $\|\cdot\|_p$-convergence, $f|_{L^\infty}$ is lsc on $(L^\infty, \|\cdot\|_\infty)$ and by dilatation monotonicity we obtain

$$f(X) \le \liminf_{n\to\infty} f(Y_n) \le f(Y).$$

Interchanging the role of X and Y in the argument above also yields $f(Y) \le f(X)$, so $f(X) = f(Y)$. Finally, if $p < \infty$, let $X, Y \in L^p$ with $X \overset{d}{=} Y$. Consider the partition $A_i^n := (i/n, (i+1)/n]$, $i = -n^2, \ldots, n^2 - 1$, and $A_{n^2}^n := (n, \infty)$, $A_{-n^2-1}^n := (-\infty, -n]$ of $\mathbb{R}$ and let $B_i := \{X \in A_i\}$ and $C_i := \{Y \in A_i\}$. Also let $\mathcal{A}_n := \sigma(B_i \mid i = -n^2 - 1, \ldots, n^2)$ and $\mathcal{G}_n := \sigma(C_i \mid i = -n^2 - 1, \ldots, n^2)$. Then for each $n \in \mathbb{N}$ we have $E[X \mid \mathcal{A}_n] \overset{d}{=} E[Y \mid \mathcal{G}_n]$ and the sequences of bounded random variables $E[X \mid \mathcal{A}_n]$ and $E[Y \mid \mathcal{G}_n]$ converge in L^p to X and Y respectively. By lsc and dilatation monotonicity of f we obtain:

$$f(X) \le \liminf_{n\to\infty} f(E[X \mid \mathcal{A}_n]) \le f(X),$$

so

$$f(X) = \lim_{n\to\infty} f(E[X \mid \mathcal{A}_n])$$

and similarly

$$f(Y) = \lim_{n\to\infty} f(E[Y \mid \mathcal{G}_n]).$$

Moreover, the already proved law invariance of f on L^∞ finally yields

$$f(X) = \lim_{n\to\infty} f(E[X \mid \mathcal{A}_n]) = \lim_{n\to\infty} f(E[Y \mid \mathcal{G}_n]) = f(Y).$$

□

Note that the proof of Theorem 2.1, in particular the part where we refer to [3], relies on a frictionless probability space in the sense that it be atomless.

Remark 2.2. *Indeed, (iii) $\Rightarrow$ (i) in the proof of Theorem 2.1 shows that a lsc dilatation monotone function $f : L^p \to (-\infty, \infty]$ is law invariant, i.e. in this direction we may drop the convexity assumption. However, the converse is not true as $\delta(\cdot \mid \{Y \mid \mathrm{law}(Y) = \mu\})$ for a non-degenerate probability distribution μ with finite p-th moment is a lsc ($\{Y \mid \mathrm{law}(Y) = \mu\}$ is closed) and law invariant function on L^p which is not dilatation monotone (and neither $\succeq$-monotone).* ◇

For a probability distribution μ on $\mathbb{R}$ with finite first moment let

$$M(\mu) = \{X \in L^1 \mid \mathrm{law}(X) = \mu\} \quad \text{and} \quad C(\mu) = \{X \in L^1 \mid \mathrm{law}(X) \preceq \mu\}.$$

The fact that the convex order $\succeq$ is indeed an order on the probability distributions of random variables clarifies the definition of $C(\mu)$. Ryff [6] and later also Ekeland and Schachermayer [5, Theorem 16] show that $C(\mu) = \overline{co}M(\mu)$ where $\overline{co}M(\mu)$ denotes the closed convex hull of $M(\mu)$ and the closure is taken with respect to the 1-norm $\|\cdot\|_1$ in L^1. Moreover, Dana shows in [3, Lemma 2.3] that $D(\mu) = C(\mu) + L^1_+$ where

$$D(\mu) := \{X \in L^1 \mid \text{law}(X) \preceq_{dc} \mu\}.$$

Based on Theorem 2.1 we are able to give easy alternative proofs of these results and add yet some other characterizations. To this end let

$$\mathcal{E}(Y) := \{E[\ldots E[Y \mid \mathcal{G}_1] \ldots \mid \mathcal{G}_K] \mid \mathcal{G}_k,\ k = 1, \ldots, K, \text{ are sub-}\sigma\text{-algebras of } \mathcal{F},\ K \in \mathbb{N}\}$$

be the smallest dilatation monotone set containing $Y \in L^1$. Note that the closures in the following theorem are in $(L^1, \|\cdot\|_1)$.

Theorem 2.3. *(i)* $C(\mu) = \overline{\text{co}}\, M(\mu) = \overline{\text{co}}\, \mathcal{E}(Y)$ *for every random variable* Y *such that* $\text{law}(Y) = \mu$*.*

(ii) $D(\mu) = C(\mu) + L^1_+ = \overline{\text{co}}(M(\mu) + L^1_+) = \overline{\text{co}}(\mathcal{E}(Y) + L^1_+)$ *for every random variable* Y *such that* $\text{law}(Y) = \mu$*.*

Proof. (i): First equality: It is readily verified that $C(\mu)$ is closed and convex in L^1. As $M(\mu) \subset C(\mu)$ we must therefore have $\overline{co}M(\mu) \subset C(\mu)$. The converse inclusion can be derived by realizing that the indicator function $\delta(\cdot \mid \overline{co}M(\mu))$ is convex, lsc, and law invariant. The law invariance of the set $\overline{co}M(\mu)$ follows from Lemma 2.4 below. Thus Theorem 2.1 implies that $\delta(\cdot \mid \overline{co}M(\mu))$ is $\succeq$-monotone. Hence, $\delta(X \mid \overline{co}M(\mu)) \le \delta(Y \mid \overline{co}M(\mu)) = 0$ whenever $X \preceq Y$ and Y has the distribution μ. This in turn means that $X \in \overline{co}M(\mu)$ and so $C(\mu) \subset \overline{co}M(\mu)$ follows.

Second equality: On the one hand, clearly $\mathcal{E}(Y) \subset C(\mu)$, so $\overline{\text{co}}\, \mathcal{E}(Y) \subset C(\mu)$. On the other hand, $\delta(\cdot \mid \overline{\text{co}}\, \mathcal{E}(Y))$ is a lsc convex and dilatation monotone function. Hence it is law invariant and preserving the convex order according to Theorem 2.1. Thus the same argument as in the proof of $C(\mu) \subset \overline{co}M(\mu)$ yields $C(\mu) \subset \overline{\text{co}}\, \mathcal{E}(Y)$.

(ii): First we show

$$\overline{\text{co}}(M(\mu) + L^1_+) = C(\mu) + L^1_+. \tag{2.2}$$

The inclusion $C(\mu) + L^1_+ \subset \overline{\text{co}}(M(\mu) + L^1_+)$ is clear from (i). For the converse inclusion note that $C(\mu) = \overline{\text{co}}\, M(\mu)$ is weakly compact (i.e. compact in the $\sigma(L^1, L^\infty)$ topology). Indeed, for every $X \in M(\mu)$ and $m \ge 0$ we have that

$$E[|X| 1_{\{|X| \ge m\}}] = \int |x| 1_{\{|x| \ge m\}} d\mu(x)$$

which does not depend on X and thus shows that $M(\mu)$ is uniformly integrable. Hence, $M(\mu)$ is relatively weakly compact by the Dunford-Pettis theorem ([4], Theorem IV.8.9), and so is its convex hull according to the Krein-Smulian theorem ([1], Theorem 10.15). Since any convex subset of a Banach space (in this case $(L^1, \|\cdot\|_1)$) is weakly closed if and only if it is norm closed, $C(\mu) = \overline{\text{co}}\, M(\mu)$ is weakly compact. In this respect we also recall the Eberlein-Smulian theorem ([1], Theorem 10.13) which states that for a subset of a Banach space weakly sequentially compactness is equivalent to relative weak compactness. Now we show that $C(\mu)+L^1_+$ is closed: Let $(X_n)_{n\in\mathbb{N}}\subset C(\mu)+L^1_+$ be a sequence which converges to X in $(L^1,\|\cdot\|_1)$. Then for every n we find Y_n and P_n such that $X_n=Y_n+P_n$ and $Y_n\in C(\mu)$ and $P_n\in L^1_+$. As $C(\mu)$ is weakly sequentially compact in $(L^1,\sigma(L^1,L^\infty))$ there is a subsequence of $(Y_n)_{n\in\mathbb{N}}$ which we without loss of generality also denote by $(Y_n)_{n\in\mathbb{N}}$ which converges weakly to some $Y\in C(\mu)$. Hence $P_n=X_n-Y_n$ converges weakly to some $P\in L^1$ such that $X=Y+P$, and as in particular $0\le E[P_n 1_{\{P<0\}}]\to E[P1_{\{P<0\}}]$ we infer that $P\ge 0$. Hence, $C(\mu)+L^1_+$ is closed. Since $C(\mu)+L^1_+$ is also convex and $M(\mu)\subset C(\mu)$ we verify (2.2). In the same way we deduce that $\overline{\mathrm{co}}(\mathcal{E}(Y)+L^1_+)=C(\mu)+L^1_+$ which proves the third equality. Since $M(\mu)+L^1_+$ is a law invariant set, applying Lemma 2.4 below and (2.2), we see that $\overline{\mathrm{co}}(M(\mu)+L^1_+)$ is a convex closed law invariant set with the property that $X\in\overline{\mathrm{co}}(M(\mu)+L^1_+)$ and $Y\ge X$ implies $Y\in\overline{\mathrm{co}}(M(\mu)+L^1_+)$. Consequently $\delta(\cdot\mid\overline{co}(M(\mu)+L^1_+))$ is convex, lsc, law invariant, and antitione with respect to the $\mathbb{P}$-a.s. order. By Theorem 2.1 we deduce that $\delta(\cdot\mid\overline{co}(M(\mu)+L^1_+))$ is $\succeq_{dc}$-monotone. This implies $D(\mu)\subset\overline{\mathrm{co}}(M(\mu)+L^1_+)$. Now as $D(\mu)$ is easily verified to be convex and closed and as $M(\mu)+L^1_+\subset D(\mu)$, we also have the opposite inclusion which finally yields $D(\mu)=\overline{\mathrm{co}}(M(\mu)+L^1_+)$. □

Lemma 2.4. *Let $C\subset L^p$, $p\in[1,\infty]$, be a law invariant set, i.e. $X\in C$ and* $\mathrm{law}(Y)=\mathrm{law}(X)$ *implies $Y\in C$. Then:*

(i) The closure $\overline{C}$ of C in $(L^p,\|\cdot\|_p)$ is law invariant.

(ii) The closed convex hull $\overline{\mathrm{co}}\,C$ of C in $(L^p,\|\cdot\|_p)$ is law invariant.

Proof. (i): The first assertion follows in the same way as the second step in the proof of (ii), so we omit it.
(ii): Suppose that $X\stackrel{d}{=}\sum_{i=1}^n\alpha_iY_i$ for some $Y_i\in C$. Consider the partition $A_k:=\left(\frac{k}{2^m},\frac{k+1}{2^m}\right]$, $k\in\mathbb{Z}$, of $\mathbb{R}$ and let $B_k:=\{X\in A_k\}$ and $C_k:=\{\sum_{i=1}^n\alpha_iY_i\in A_k\}$. Since the probability space is standard and since $\mathbb{P}(B_k)=\mathbb{P}(C_k)$ there is a measure preserving transformation $\pi:\Omega\to\Omega$ such that $\pi(B_k)=C_k$ and $\pi(C_k)=B_k$ for all k. Let $\widetilde{Y}_i:=Y_i\circ\pi$ for $i=1,\dots,n$. Then the random vectors $(\widetilde{Y}_1,\dots,\widetilde{Y}_n)$ and $(Y_1,\dots,Y_n)$ are identically distributed. Hence, $X_k:=\sum_{i=1}^n\alpha_i\widetilde{Y}_i\stackrel{d}{=}\sum_{i=1}^n\alpha_iY_i$ and also $X_k\in\mathrm{co}\,C$ because $\widetilde{Y}_i\stackrel{d}{=}Y_i$. Moreover, we have that $\|X-X_k\|_\infty\le\frac{1}{2^m}$. By letting $m\to\infty$ we infer that $X\in\overline{\mathrm{co}}\,C$.

If $X\stackrel{d}{=}Y=\lim_{n\to\infty}Y_n$ for a sequence $(Y_n)_{n\in\mathbb{N}}\in\mathrm{co}\,C$, then, as above (case $n=1$), there is for any $\epsilon>0$ a measure preserving transformation $\pi:\Omega\to\Omega$ such that $\|X-Y\circ\pi\|_\infty\le\epsilon/2$. As $Y\circ\pi=\lim_{n\to\infty}Y_n\circ\pi$, we may find an $m\in\mathbb{N}$ such that $\|X-Y_m\circ\pi\|_p\le\epsilon$ where $Y_m\in\overline{\mathrm{co}}\,C$ according to the first step. □

If μ has finite p-th moment for $p\in[1,\infty)$, then $M(\mu)\subset L^p$ and also $C(\mu)\subset L^p$ as $x\mapsto|x|^p$ is convex. Since $\nu\preceq\mu$ implies that the support of ν is contained in the closed convex hull of the support of μ (otherwise convex functions which take positive values outside the closed convex hull of the support of μ and 0 else would yield a contradiction) we observe that if μ has a compact support, then $C(\mu)\subset L^\infty$. Moreover, as Theorem 2.1 and Lemma 2.4 hold for any $p\in[1,\infty]$, we infer by the same arguments as in the the proof of Theorem 2.3 (i) that if there is $Y\in L^p$ with distribution μ, then the closed convex hull of $M(\mu)$ or $\mathcal{E}(Y)$ in $(L^p,\|\cdot\|_p)$ coincides with the closed convex hull of $M(\mu)$ or $\mathcal{E}(Y)$ in $(L^1,\|\cdot\|_1)$, respectively, namely with $C(\mu)$. Hence, the deduction of the following corollary is straightforward:

Corollary 2.5. *Let $Y\in L^p$, $p\in[1,\infty]$, have distribution μ. Then* *(i)* $C(\mu) = \overline{\text{co}}\, M(\mu) = \overline{\text{co}}\, \mathcal{E}(Y)$ *where the closures are taken in* $(L^p, \|\cdot\|_p)$.

(ii) $D(\mu) \cap L^p = C(\mu) + L^p_+ = \overline{\text{co}}(M(\mu) + L^p_+) = \overline{\text{co}}(\mathcal{E}(Y) + L^p_+)$ *where the closures are taken in* $(L^p, \|\cdot\|_p)$.

As an immediate consequence of Corollary 2.5 we obtain that $X \succeq_{dc} Y$ (i.e. $X \preceq_{uni} Y$) if and only if there exists $\widetilde{Y} \in L^1$ and $P \in L^1_+$ such that $X \succeq \widetilde{Y}$ and $Y = \widetilde{Y} + P$. If $X \in L^p$, $p \in [1, \infty]$, then $\widetilde{Y}$ in L^p and if also $Y \in L^p$, then also $P \in L^p_+$. This in turn implies the well-known relation that $X \succeq Y$ if and only if $X \succeq_{dc} Y$ and $E[X] = E[Y]$. The 'only if' part is clear by (2.1) and since $x \mapsto x$ and $x \mapsto -x$ are convex functions. In order to show the 'if' part decompose $Y = \widetilde{Y} + P$ as above. As $E[X] = E[Y]$ and thus $E[Y] = E[\widetilde{Y}]$, we conclude that $P = 0$, so $Y = \widetilde{Y}$.

Let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{F}$. We write

$$M^{\mathcal{A}}(\mu) = \{Y \in M(\mu) \mid Y \text{ is } \mathcal{A}\text{-measurable}\} \text{ and } C^{\mathcal{A}}(\mu) = \{Y \in C(\mu) \mid Y \text{ is } \mathcal{A}\text{-measurable}\}.$$

Lemma 2.6. *Suppose there is* $X \in L^p$, $p \in [1, \infty]$, *with distribution* μ. *Let* $\mathcal{A}$ *and* $\mathcal{G}$ *be independent sub-*σ*-algebras of* $\mathcal{F}$ *such that both* $(\Omega, \mathcal{A}, \mathbb{P})$ *and* $(\Omega, \mathcal{G}, \mathbb{P})$ *are non-atomic. Then*

$$C^{\mathcal{A}}(\mu) = \overline{\{E[Y \mid \mathcal{A}] \mid Y \in M(\mu)]\}}$$

where the closure is taken in $(L^p, \|\cdot\|_p)$.

Proof. According to Corollary 2.5 we know that $C^{\mathcal{A}}(\mu) = \overline{\text{co}}\, M^{\mathcal{A}}(\mu)$. Let $Y_1, \dots Y_n \in M^{\mathcal{A}}(\mu)$ and $\alpha_i > 0$, $i = 1, \dots, n$, such that $\sum_i \alpha_i = 1$. Choose disjoint sets $A_i \in \mathcal{G}$ with $\mathbb{P}(A_i) = \alpha_i$ and $\bigcup A_i = \Omega$. Then

$$\sum_i \alpha_i Y_i = E[\sum_i Y_i 1_{A_i} \mid \mathcal{A}]$$

because $\mathcal{A}$ and $\mathcal{G}$ are independent. Moreover, $\text{law}(\sum_i Y_i 1_{A_i}) = \mu$. Therefore, it follows that

$$\text{co}\, M^{\mathcal{A}}(\mu) \subset \{E[Y \mid \mathcal{A}] \mid Y \in M(\mu)\} \quad \text{and so} \quad C^{\mathcal{A}}(\mu) \subset \overline{\{E[Y \mid \mathcal{A}] \mid Y \in M(\mu)\}}.$$

Conversely, we have that $\{E[Y \mid \mathcal{A}] \mid Y \in M(\mu)]\} \subset C^{\mathcal{A}}(\mu)$ since $E[Y \mid \mathcal{A}]$ is $\mathcal{A}$-measurable and $\text{law}(E[Y \mid \mathcal{A}]) \preceq \mu$. Thus the assertion follows. □

Based on Lemma 2.6 the following theorem now establishes the connection between dilatation monotonicity and $\succeq$- or $\succeq_{dc}$-monotonicity without any convexity assumption on the underlying function.

Theorem 2.7. *Let* $f : L^p \to (-\infty, \infty]$, $p \in [1, \infty]$, *be a lsc function. Then the following are equivalent:*

(i) f *is* $\succeq$*-monotone.*

(ii) f *is dilatation monotone.*

Also the following are equivalent:

(iii) f *is antitone with respect to the* $\mathbb{P}$*-a.s. order and* $\succeq$*-monotone.*

(iv) f *is antitone with respect to the* $\mathbb{P}$*-a.s. order and dilatation monotone.*

(v) f is $\succeq_{dc}$-monotone.

If any of the conditions (i) – (v) is satisfied, then f is law invariant.

Proof. $(i) \Leftrightarrow (ii)$: $(i) \Rightarrow (ii)$ is obvious, so we only have to prove that dilatation monotonicity implies that f preserves $\succeq$. To this end let $Y \succeq X$ and choose $\mathcal{A}$ and $\mathcal{G}$ as in Lemma 2.6 and let $\widetilde{X}$ be a $\mathcal{A}$-measurable random variable such that $\widetilde{X} \overset{d}{=} X$. Then $\widetilde{X} \in C^{\mathcal{A}}(\mathrm{law}(Y))$ and Lemma 2.6 implies that there is (a probably trivial) sequence of conditional expectations $E[Y_n \mid \mathcal{A}]$ converging to $\widetilde{X}$ such that $Y_n \in M(\mathrm{law}(Y))$ for all n. Hence, by law invariance (Remark 2.2), lsc, and dilatation monotonicity of f we conclude that

$$f(X) = f(\widetilde{X}) \leq \liminf_{n\to\infty} f(E[Y_n \mid \mathcal{A}]) \leq \liminf_{n\to\infty} f(Y_n) = f(Y).$$

$(iii) \Leftrightarrow (iv) \Leftrightarrow (v)$: $(iii) \Leftrightarrow (iv)$ follows from $(i) \Leftrightarrow (ii)$, and $(v) \Rightarrow (iii)$ from the fact that both $X \leq Y$ or $X \succeq Y$ imply $X \succeq_{dc} Y$. Finally, in order to show that $(iii) \Rightarrow (v)$ let $X \succeq_{dc} Y$. As above we decompose $Y = \widetilde{Y} + P$ where $\widetilde{Y} \preceq X$ and $P \in L^p_+$ (Corollary 2.5). Then

$$f(Y) = f(\widetilde{Y} + P) \leq f(\widetilde{Y}) \leq f(X).$$

□

Example 2.8. *Let μ and ν be two distributions such that neither $\mu \succeq \nu$ nor $\nu \succeq \mu$. Then $\delta(\cdot \mid C(\mu) \cup C(\nu))$ is a lsc., law invariant, and dilatation monotone function which is of course also $\succeq$-monotone, but which is not convex since $C(\mu) \cup C(\nu)$ is not convex.* ◇

Example 2.9. *For a non-degenerate random variable $Y \in L^p$, the function $\delta(\cdot \mid \mathcal{E}(Y))$ is dilatation monotone, but not law invariant, and thus not preserving the convex order. The point is that $\mathcal{E}(Y)$ is not closed.* ◇

Since for every closed set $A \subset L^p$ the indicator function $\delta(\cdot \mid A)$ is lsc, we arrive at the following version of Theorem 2.7 for sets.

Corollary 2.10. *Let $A \subset L^p$, $p \in [1, \infty]$, be a closed set. Then the following are equivalent:*

(i) For all $Y \in A$ we have that $C(\mathrm{law}(Y)) \subset A$.

(ii) For all $Y \in A$ and any sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$ we have that $E[Y \mid \mathcal{G}] \in A$.

Also the following are equivalent:

(iii) Property (i) holds and $A + L^p_+ = A$.

(iv) Property (ii) holds and $A + L^p_+ = A$.

(v) For all $Y \in A$ we have that $(D(\mathrm{law}(Y)) \cap L^p) \subset A$.

If any of the conditions (i) – (v) is satisfied, then A is law invariant.

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