Sensitivity of life insurance reserves via Markov semigroups

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Contents

Why sensitivities?

- Risk management means managing the available capital (CFO and CRO responsibility)
- Regulatory capital requirements (e.g., standard model in Solvency II) are based on parameter scenarios
- Thus, sensitivities of insurance reserves with respect to the valuation basis are of particular interest
- Understanding the sensitivities is key to premium calculation, reserving and ultimately the survival of the insurer

Thiele's differential equation I

- The reserve is usually defined as the discounted expected benefit payments less than the discounted expected premium payments (equivalence principle)
- In 1875 Thiele devised a differential equation (and difference equation) for the evolution of the reserve

$$\frac{d}{dt}V_t = \pi_t - b_t\mu_{x+t} + (r + \mu_{x+t})V_t.$$

- Unification of the time discrete and continuous case in a stochastic integral equation [MS97]

Thiele's differential equation II

- Combination with developments in financial mathematics (Black-Scholes, term-structure models, ...) leads to generalized Thiele equations, cf. [Nor91] or [Ste06]
- These generalizations model modern life insurance products whose benefits explicitly depend on capital markets
- In product design and capital requirements current attention shifts towards worst-case analyses with respect to the valuation basis
- Examples include [Chr11b], [Chr11a], [Chr10] and [CS11]

Our model life insurance contract

- We consider a multi-state life insurance policy with distribution of a surplus as in [Ste07]
- The surplus can be invested in a risk-free asset and a risky asset, the latter being modelled by an Itô process
- The reserve satisfies a system of partial differential equations (PDEs)
- Objective: solve the PDEs by semigroup techniques and then assess sensitivities

The aim is to investigate the Thiele PDE using linear operators

- Basic idea is to express economic forces by linear operators
- Motivation: quantum mechanics and operator algebras
- Key results
  - Uniform continuity of the reserve with respect to financial, mortality and payment assumptions
  - Pointwise bounds on the gradient of the reserve as a function of the surplus
  - Factorization of the reserve into risk types (financial, insurance, payment)
- Basis for treatment of polynomial processes (including Lévy and affine processes)

Selected other approaches to sensitivities

Key ideas from the literature

- Valuation basis depends on a single parameter θ. Differentiate Thiele's equation with respect to θ and solve ensuing PDE. Cf. [KN03]
- Valuation basis lives in a Hilbert space. Consider the reserve as a functional of the valuation basis and apply Fréchet derivative with respect to valuation basis. Cf. [Chr08]

Contents

Start with the reserve as a conditional expectation

Consider a life insurance policy with benefit payments that depend on a surplus. The surplus can be invested in a risk-free and a risky asset. All processes are (jointly) Markov:

- Z_t: process with values in $\{1, \ldots, n\}$, state of the insured person
- X_t: process for the value of the surplus (SDE)
- B_t: process for benefit payments
- D_t: process for dividend payments from the surplus

Define the *market reserve* V^j of the contract in state j as

$$V^j(t,x) = \mathbb{E}\left[\int_t^T e^{(s-t)r} d(B+D)(s) \middle| Z(t) = j, X(t) = x\right],$$

with the policy terminating at time T

The reserve satisfies a PDE system I

The reserve vector $\mathbf{V} = (V^1, \ldots, V^n)^\top$ satisfies

$$\left.\begin{array}{rcl} 0 & = & \partial_t V^j(t,x) + \mathcal{D}^j(t) V^j(t,x) + \beta^j(t,x) - rV^j(t,x), \\ 0 & = & V^j(T,x), \end{array}\right\}$$

on $[0, T] \times \mathbb{R}$ where

$$\begin{aligned} \mathcal{D}^j(t) =& \tfrac{1}{2}\pi(t,x)^2\sigma^2 x^2 \partial_x^2 + \left(rx + c^j(t) - \delta^j(t,x)\right)\partial_x \\ &+ \sum_{k \neq j} \mu^{jk}(t)\left(V^k(t, x + c^{jk}(t) - \delta^{jk}(t,x)) - V^j(t,x)\right), \\ \beta^j(t,x) =& b^j(t) + \delta^j(t,x) + \sum_{k \neq j} \mu^{jk}(t)\left(b^{jk}(t) + \delta^{jk}(t,x)\right). \end{aligned}$$

For the derivation of these equations see [Ste06, Ste07]

The reserve satisfies a PDE system II

Meaning of the variables and coefficients

$$
\begin{aligned}
t &= \text{time} \\
x &= \text{value of the surplus} \\
T &= \text{maturity of the contract} \\
V^j(t,x) &= \text{reserve in state } j \\
r &= \text{constant risk-free interest rate} \\
\pi(t,x) &= \text{surplus share invested in the risky asset} \\
\sigma &= \text{diffusion coefficient for the risky asset} \\
b^{jk}(t), b^j(t) &= \text{benefit payments} \\
\mu^{jk}(t) &= \text{transition intensities} \\
\delta^j(t,x),\ \delta^{jk}(t,x) &= \text{dividends from the surplus} \\
c^j(t),\ c^{jk}(t) &= \text{contributions to the surplus}
\end{aligned}
$$

The reserve satisfies a PDE system III

Hypothesis

(i) *Coefficients of the differential operators*

(a) *there is a* $\pi_0 > 0$ *with* $\pi(x) \geq \pi_0$ *for all* $x \in \mathbb{R}_+$,

(b) π, c^j, δ^j *are in* $C^{\alpha/2,\alpha}_{loc}([0,T] \times \mathbb{R}_+)$ *for a* $\alpha \in (0,1)$,

(c) *the function* π *is bounded and* $c^j \geq 0$.

(ii) *Regularity of payments, dividends and intensities*

(a) b^j *and* μ^{jk} *belong to* $C([0,T])$,

(b) δ^{jk} *belongs to* $C^{0,\alpha}([0,T] \times \mathbb{R})$ *for all* j, k.

(iii) *Boundedness of dividend payments*

(a) *there is a constant* $k > 0$ *with* $0 \leq \delta^j(t,x) \leq kx$,

(b) *the term* $\delta^j(t,x)\frac{-\log x}{x(1+\log^2 x)}$ *is bounded for* $x \to 0$,

(c) *for all* x, t *we have* $x + c^{jk}(t) - \delta^{jk}(t,x) \geq 0$.

Relaxation of assumptions I

Coefficients of the differential operators

(a) there is a $\pi_0 > 0$ with $\pi(x) \geq \pi_0$ for all $x \in \mathbb{R}_+$,

(b) π, c^j, δ^j are in $C^{\alpha/2,\alpha}_{\text{loc}}([0, T] \times \mathbb{R}_+)$ for a $\alpha \in (0,1)$,

(c) the function π is bounded and $c^j \geq 0$.

- Allow for $\pi(x) \geq 0$ i.e., surrender uniform ellipticity, the PDE is degenerate
- The analysis is done by regularizing the equation i.e., one considers $\mathcal{A}^j + \epsilon\Delta$ for $\epsilon > 0$ and the Laplace operator Δ
- This leads to a solution $\mathbf{V}_\epsilon$. Now consider $\epsilon \to 0$ and show weak convergence e.g., in L^2

Relaxation of assumptions II

Regularity of payments, dividends and intensities

(a) b^j and μ^{jk} belong to $C([0, T])$,

(b) δ^{jk} belongs to $C^{0,\alpha}([0, T] \times \mathbb{R})$ for all j, k.

- Allow for measurable coefficients
- Leads to solution in L^∞ or L^p-spaces

Relaxation of assumptions III

Boundedness of dividend payments

(a) there is a constant $k > 0$ with $0 \leq \delta^j(t, x) \leq kx$,

(b) the term $\delta^j(t, x)\frac{-\log x}{x(1+\log^2 x)}$ is bounded for $x \to 0$,

(c) for all x, t we have $x + c^{jk}(t) - \delta^{jk}(t, x) \geq 0$.

- Crucial in our framework
- However, no practical limitations as the important case of δ^* linear in x is covered

Contents

Solution semigroups of PDEs I

The basic example is the heat equation on $\mathbb{R}^n$:

$$\partial_t u(t,x) = \Delta u(t,x),$$
$$u(0,x) = g(x).$$

View this as an abstract evolution equation in $C^{1,2}(\mathbb{R}^n)$ by regarding $u(t,\cdot)$ as and element of $C([0,T]; C^2(\mathbb{R}^n))$. Then rewrite the PDE as

$$\partial_t u(t) = \Delta u(t),$$
$$u(0) = g,$$

a first-order ordinary differential equation in t. Formally solve this as

$$u(t) = e^{t\Delta} u(0).$$

Does this make sense?

Properties of semigroups

Operator families of the type e^{tA} acting on a Banach space X, should have the following properties:

(i) $e^{tA}e^{sA} = e^{(t+s)A}$ for $t, s > 0$ (semigroup property)

(ii) $\lim_{t\to 0} e^{tA}x = x$ for all $x \in X$ (strong continuity)

(iii) $\partial_t a^{tA} = Ae^{tA} = e^{tA}A$ (solution of the PDE)

In our case we will not have strong continuity, still $e^{0A} = id$.
See also [Ama95], [Paz83], [EN00], [Lun95] for the construction of semigroups and their application to PDEs

The generator of a semigroup

Let $T(t)$ be a strongly continuous semigroup acting on a Banach space X. Define

$$D(A) = \left\{ f \in X : \frac{T(t)f - f}{t} \text{ converges in norm for } t \to 0_+ \right\}$$

and set

$$A(f) = \lim_{t \to 0_+} \frac{T(t)f - f}{t} \text{ for } f \in D(A).$$

We call A the *(infinitesimal) generator* of the semigroup and $D(A)$ the *domain* of A. $D(A)$ is a linear subspace of X and A is a linear map $D(A) \to X$. Usually, $D(A)$ is very hard to identify precisely

Construction of semigroups

There are several ways to construct operators e^{tA}:

- as the solution to $\partial_t u = Au$
- as a Taylor series in case $A : X \to X$ is bounded

$$e^{tA} = I + tA + \frac{1}{2}t^2A^2 + \cdots$$

- by functional calculus on a Banach algebra
- by a Cauchy integral, if the resolvent $(\lambda - A)^{-1}$ is bounded

$$e^{tA} = \frac{i}{2\pi}\int_\Gamma e^{\lambda t}(\lambda - A)^{-1} d\lambda.$$

- by the theorems of Hille-Yosida, Ray, Phillipps, etc.

Relationship with stochastic processes

Morally: let $(X_t)_{t\geq 0}$ be a stochastic process with state space $\mathbb{R}^n$. Fix $x \in \mathbb{R}^n$. The semigroup $T(t)$ for X_t acts on functions $u : \mathbb{R}^n \to \mathbb{R}$ as follows

$$[T(t)u](x) = \mathbb{E}^x(u(X_t)).$$

The generator A is given by

$$Au = \lim_{t\to 0} \frac{T(t)u - u}{t}.$$

In case of a Brownian motion with drift A has the form

$$A = \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x) \frac{\partial}{\partial x_i}.$$

Generalization to evolution families

So far we had time-independent (autonomous) generators. Now generalization to time-dependence which leads to *evolution families*. A family of linear operators $\{G(t,s) : 0 \leq s \leq t \leq T\}$ in $\mathcal{B}(X)$ is called *evolution family* if

(i) $G(t,s)G(s,r) = G(t,r)$ for $0 \leq r \leq s \leq t \leq T$ and $G(s,s) = id$

(ii) $G(t,s)$ maps X to D with D the domain of $A(t)$, where we assume that all $A(t)$ have the same domain

(iii) The map $t \mapsto G(t,s)$ is differentiable on $(s,T]$ with values in $\mathcal{B}(X)$ and for $0 \leq s \leq t \leq T$ we have $\partial_t G(t,s) = A(t)G(t,s) = G(t,s)A(t)$.

Contents

Consider Thiele as an abstract evolution equation I

Step 1: define a time-dependent linear operator $\mathbf{T} = \mathbf{T}(t)$ acting on $C_b([0, T] \times \mathbb{R}) \otimes \mathbb{R}^n$ by

$$\mathbf{T} = \begin{pmatrix} -\sum_{k \neq 1} \mu^{1k} 1 & \mu^{12} T^{12} & \mu^{13} T^{13} & \dots & \mu^{1n} T^{1n} \\ \mu^{21} T^{21} & -\sum_{k \neq 2} \mu^{2k} 1 & \mu^{23} T^{23} & \dots & \mu^{2n} T^{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu^{n1} T^{n2} & \mu^{n2} T^{n2} & \mu^{n3} T^{n3} & \dots & -\sum_{k \neq n} \mu^{nk} 1 \end{pmatrix}$$

Here, $\mu^{jk} = \mu^{jk}(t)$ and the $T^{jk} = T^{jk}(t)$ are linear operators:

$$\left(T^{ik} f\right)(t, x) = f\left(t, x + c^{jk}(t) - \delta^{jk}(t, x)\right)$$

Morally: insurance risk expressed by $\mathbf{T}$

Consider Thiele as an abstract evolution equation II

Step 2: spacetime transformation $\tau = T - t$ and $y = \log x$. Define

$$\mathcal{A}^j = \tfrac{1}{2}\pi^2\sigma^2\partial_y^2 + \left(r + \left(c^j - \delta^j\right)e^{-y} - \tfrac{1}{2}\pi\sigma^2\right)\partial_y.$$

With the diagonal operator $\mathcal{A} = \begin{pmatrix} \mathcal{A}^1 & & \\ & \ddots & \\ & & \mathcal{A}^n \end{pmatrix}$ the reserve vector satisfies

$$\left.\begin{array}{rcl} \partial_\tau \mathbf{V} & = & \mathcal{A}(\tau)\mathbf{V} + \mathbf{TV} - r\mathbf{V} + e^{r\tau}\beta \\ \mathbf{V}(0) & = & 0, \end{array}\right\} \qquad (1)$$

an abstract initial value problem on a suitable Banach space

Formulation as an integral equation with semigroups

- Let $\mathbf{G}$ be the evolution family generated by $\mathcal{A}$ i.e., a family of linear operators $\mathbf{G}(\tau, s)$ on a suitable space such that

$$\mathbf{G}(\tau, s)\mathbf{G}(s, \rho) = \mathbf{G}(\tau, \rho)$$

  for $\rho \leq s \leq \tau$ and $\mathbf{G}(\tau, \tau) = id$
- The existence of $\mathbf{G}$ is non-trivial as $\mathcal{A}$ has exponentially growing first-order coefficients
- $\mathbf{V}$ is a *mild solution* of (1) if the Duhamel formula is satisfied

$$\mathbf{V}(\tau) = \int_0^{\tau} \mathbf{G}(\tau, s) \left[\mathbf{T}(s)\mathbf{V}(s) + e^{-r(\tau - s)}\beta(s)\right] \, ds \qquad (2)$$

Contents

The PDE has a unique mild solution I

Theorem

Assume the coefficients of $\mathcal{A}$ are in $C^{\alpha/2,1+\alpha}$ for a $\alpha \in (0,1)$. Then there is a unique mild solution $\mathbf{V}$ in the space $C^{0,\alpha}([0,T] \times \mathbb{R}) \otimes \mathbb{R}^n$.

The Duhamel decomposition (2) of $\mathbf{V}$ shows the factorization of the integrand into operators:

(i) market risks from the investment in the risky asset as represented by $\mathbf{G}$,

(ii) the effect of net payments represented by the multiplication operator β, and

(iii) insurance risk represented by $\mathbf{T}$

The PDE has a unique mild solution II

Express the solution explicitly in terms of a Neumann series (Dyson series in physics, Peano series in matrix analysis). Let $f(\tau) = \int_0^\tau e^{rs}\beta(s)\, ds$, then

$$\mathbf{V} = e^{-r\tau}\left(f + \mathbf{GT}\#f + \mathbf{GT}\#\mathbf{GT}\#f + \cdots\right) \tag{3}$$

under the operation

$$(\mathbf{GT}\#\xi)(\tau) = \int_0^\tau \mathbf{G}(\tau, s)\mathbf{T}(s)\xi(s)ds.$$

The series converges in $C^{0,\alpha}([0, T] \times \mathbb{R}) \otimes \mathbb{R}^n$. This leads to a conceptual explanation how the reserve depends on payments. The series is also an asymptotic expansion in τ and can be used to approximate $\mathbf{V}$

Continuous dependence of the reserve on the data

Theorem

Let $Y_1 = C_b(\mathbb{R}) \otimes \mathbb{R}^n$, $Y_2 = C_b([0,T] \times \mathbb{R}) \otimes \mathbb{R}^n$, $\varphi(\tau) = \int_0^\tau e^{-r(\tau - s)} ds$, *and* $\hat{T} = \sup_\tau ||\mathbf{T}(\tau)||$. *Then*

(i) *growth:* $||\mathbf{V}(\tau)||_{Y_1} \leq ||\beta||_{Y_2} \left(\hat{T} e^{c\hat{T}\tau} \int_0^\tau \varphi(s) ds + \varphi(\tau) \right)$

(ii) *dependence on payments:*

$$||\mathbf{V}_1(\tau) - \mathbf{V}_2(\tau)||_{Y_1} \leq ||\beta_1 - \beta_2||_{Y_2} \left(\hat{T} e^{\hat{T}\tau} \int_0^\tau \varphi(s) ds + \varphi(\tau) \right)$$

(iii) *dependence on insurance risk:*

$$||\mathbf{V}_1(\tau) - \mathbf{V}_2(\tau)||_{Y_1} \leq ||\beta||_{Y_2} C(\mathbf{T}_1, \mathbf{T}_2; \tau) \sup_\tau ||\mathbf{T}_1(\tau) - \mathbf{T}_2(\tau)||,$$

with $C(\mathbf{T}_1, \mathbf{T}_2; \tau) = \hat{T}_1 e^{\hat{T}_1 \tau} \int_0^\tau \int_0^s \varphi(u) du ds + e^{\hat{T}_2 \tau} \int_0^\tau \varphi(s) ds.$

One recovers the conditional expectation almost explicitly

- Recall the stochastic representation

$$\mathbf{V}^{Z(t)}(t, X(t)) = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)} d(B+D)(s) \,\middle|\, Z(t), X(t)\right]$$

- Now special case where the surplus is unchanged in transitions between states ie., $c^{jk}(t) - \delta^{jk}(t,x) \equiv 0$
- Then (2) becomes

$$\mathbf{V}(\tau, y) = \int_0^{\tau} e^{-r(\tau-s)} \left[\mathbf{G}(\tau, s) \exp \mathbf{M}(s) \beta(s)\right](y)\, ds.$$

Here $\mathbf{M}(s) = \int_0^s \mathbf{T}(s')\, ds'$ by componentwise integration

- The product of commuting operators $\mathbf{G}(\tau, s) \exp \mathbf{M}(s)$ corresponds tp the product measure $\mathbb{Q}$

Pointwise sensitivities in the special case

Define $\beta'(s, y) = e^{rs}\beta(s) \exp \mathbf{M}(s)$.

Theorem

Choose $p > 1$ and let $W^j(\tau, y)$ be a solution of the PDE

$$\partial_\tau W^j = \mathcal{A}^j(\tau)W^j - (r - \sigma_p)W^j + \left|T^{1-1/p}\partial_y \beta'^j(\tau, \cdot)\right|^p$$
$$W^j(0) = 0,$$

where σ_p is a constant depending on $\mathcal{A}^j$. The the gradient of the reserve is bounded pointwise

$$|\partial_y V^j(\tau, y)|^p \leq W^j(\tau, y)$$

for $(\tau, y) \in [0, T] \times \mathbb{R}$

Contents

Existence via operator algebra I

Proposition

Let $\theta \in [0,1]$. *Then* $\mathbf{T}$ *is a bounded linear operator mapping* $C^{0,\theta}([0,T] \times \mathbb{R}) \otimes \mathbb{R}^n$ *to itself.*

Moreover there exists an evolution family $\mathbf{G}$ which is smoothing

Existence via operator algebra II

Proposition ([Lor11])

Each operator $\mathcal{A}^j$ *generates an evolution system* $G^j(\tau, s)$ *of bounded linear operators such that for every* $0 \leq \alpha \leq \gamma \leq 1$ *there is a constant* c *with*

$$||G^j(\tau, s)f||_{C_b^\gamma(\mathbb{R})} \leq c(\tau - s)^{-(\gamma-\alpha)/2}||f||_{C_b^\alpha(\mathbb{R})}$$

for $f \in C_b^\alpha(\mathbb{R})$ *and every* $s \leq \tau \leq T$

Existence via operator algebra III

Idea for showing existence and uniqueness of solutions:

(i) a-priori estimates via Gronwall's inequality

(ii) Explicit construction of a solution through a Neumann series, it converges by the a-priori estimates

(iii) Uniqueness of the solution again by Gronwall

Existence via operator algebra IV

Lemma (Gronwall's inequality)

Suppose that for a non-negative absolutely continuous function η on $[0, T]$

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t).$$

Then

$$\eta(t) \leq e^{\int_0^t \phi(s)ds} \left[\eta(0) + \int_0^t \psi(s)ds \right].$$

Proof: A calculation shows

$$\begin{aligned} \frac{d}{ds}\left(\eta(s)e^{-\int_0^s \phi(r)dr}\right) &= e^{-\int_0^s \phi(r)dr}\left(\eta'(s) - \phi(s)\eta(s)\right) \\ &\leq e^{\int_0^s \phi(r)dr}\psi(s), \end{aligned}$$

whence the assertion.

Existence via operator algebra V

Let $Y_1 = C^{\alpha}(\mathbb{R}) \otimes \mathbb{R}^n$ and $Y_2 = C^{0,\alpha}([0,T] \times \mathbb{R}) \otimes \mathbb{R}^n$. Uniform estimates yield

$$||\mathbf{V}(\tau)||_{Y_1} \leq c\hat{T} \int_0^{\tau} ||\mathbf{V}(s)||_{Y_1} + c||\beta||_{C^{0,\alpha}([0,T]\times\mathbb{R})\otimes\mathbb{R}^n}\varphi(\tau),$$

with $\hat{T} = \sup_s ||T(s)||$, the supremum of the operator norms of T and $\varphi(\tau) = \int_0^{\tau} e^{-r(\tau-s)}ds$. The constant c comes from Proposition 6. Gronwall now implies

$$||\mathbf{V}(\tau)||_{Y_1} \leq c||\beta||_{Y_2} \left(c\hat{T}e^{c\hat{T}\tau} \int_0^{\tau} \varphi(s)ds + \varphi(\tau) \right). \qquad (4)$$

Gives a-priori estimates: uniform norm of the reserve is bounded by global constants

Existence via operator algebra VI

Existence: General approach to solving *Volterra equations* of the form

$$u(\tau) = f(\tau) + \int_0^\tau T(s)u(s)ds,$$

cf. [Kre99]: set $Au = \int_0^\tau T(s)u(s)ds$ and write

$$u = Au + f \quad \text{or} \quad (I - A)u = f.$$

The idea is then to invert the operator $I - A$ as

$$(I - A)^{-1} = 1 + A + A^2 + \cdots .$$

Application of this Neumann series in spectral theory of Banach algebras, PDEs, etc.

Existence via operator algebra VII

Use this to find a solution for small values of T in a Neumann series with $f(\tau) = \int_0^\tau e^{rs}\beta(s)\,ds$ as

$$\mathbf{V} = e^{-r\tau}\left(f + \mathbf{GT}\#f + \mathbf{GT}\#\mathbf{GT}\#f + \cdots\right).$$

under the operation

$$(\mathbf{GT}\#\xi)(\tau) = \int_0^\tau \mathbf{G}(\tau, s)\mathbf{T}(s)\xi(s)ds.$$

Now iterate on the time axis with new initial conditions. This works because the a-priori estimates (4) only depend on global constants (and not on the time T)

Existence via operator algebra VIII

Uniqueness: Let $\mathbf{V}_1$ and $\mathbf{V}_2$ be two solutions of the PDE. Then consider $\mathbf{U} = \mathbf{V}_1 - \mathbf{V}_2$, which satisfies a linear homogeneous integral equation with initial condition $\mathbf{U}(0) = \mathbf{V}_1(0) - \mathbf{V}_2(0) = 0$:

$$\mathbf{U}(\tau) = \int_0^\tau \mathbf{G}(\tau, s)\mathbf{T}(s)\mathbf{U}(s)\, ds.$$

Gronwall then shows that $\mathbf{U}(\tau) = 0$ for all τ

Proof of the sensitivities I

This also follows from operator estimates
Uniform estimates: Illustration for the dependence on $\mathbf{T}$: consider the PDEs for $\mathbf{V}_1$ and $\mathbf{V}_2$ for two values of the operator $\mathbf{T}_1$ and $\mathbf{T}_2$. By linearity

$$\begin{aligned}\mathbf{V}_1(\tau) - \mathbf{V}_2(\tau) &= \int_0^\tau \mathbf{G}(\tau, s) \left(\mathbf{T}_1(s)\mathbf{V}_1(s) - \mathbf{T}_2(s)\mathbf{V}_2(s) \right) ds \\ &= \int_0^\tau \mathbf{G}(\tau, s)\mathbf{T}_2(s) \left(\mathbf{V}_1(s) - \mathbf{V}_2(s) \right) ds \\ &\quad + \int_0^\tau \mathbf{G}(\tau, s) \left(\mathbf{T}_1(s) - \mathbf{T}_2(s) \right) \mathbf{V}_1(s) ds.\end{aligned}$$

Then apply Gronwall twice to obtain the result.

Proof of the sensitivities II

Pointwise estimates: The basis is the

Theorem ([KLL10])

For every $p > 1$ we have for all $f \in C_b^1(\mathbb{R})$ that

$$|\left(\partial_x G^j(\tau,s)f\right)(x)|^p \leq e^{c(\tau-s)}\left(G^j(\tau,s)|\partial_x f|^p\right)(x) \tag{5}$$

with $s \leq \tau$ and $x \in \mathbb{R}$. Here c is a constant depending on p and $\mathcal{A}$

Proof of the sensitivities III

Application to

$$\mathbf{V}(\tau, y) = \int_0^{\tau} e^{-r(\tau-s)} \left[\mathbf{G}(\tau, s) \exp \mathbf{M}(s)\beta(s)\right](y)\, ds$$

leads to an integral equation whose upper bound (5) can be translated to the PDE

$$\partial_\tau W^j = \mathcal{A}^j(\tau) W^j - (r-c) W^j + \left| T^{1-1/p} \partial_y \beta'^j(\tau, \cdot)\right|^p$$
$$W^j(0) = 0.$$

This is possible as $\mathbf{V}$ is a classical solution i.e., belongs to $C^{1,2}$

Contents

Potential next steps I

Polynomial processes

- Model the risky asset by polynomial processes e.g., a Lévy process
- The operator approach can be applied formally. The operators $\mathcal{A}^j$ become pseudo-differential operators which are defined by Fourier analysis (a is the symbol of the process)

$$\mathcal{A}u(x) = \int\int e^{i\xi(x-y)} a(x,y,\xi)u(y)dyd\xi,$$

  precise structure of a from Lévy-Khinchin
- Increased technical requirements and solution living in Sobolev spaces or C^∞

Potential next steps II

Heat kernel methods

- Short-time asymptotic expansion of the reserve in τ around maturity T
- Basis is an asymptotic expansion of the Schwartz kernel of $\mathbf{G} \sim \mathbf{G}_0 + (\tau - s)\mathbf{G}_1 + \cdots$, the so-called heat kernel. This is standard in differential geometry (Atiyah-Singer index theorem), quantum gravity, financial maths, ...
- Looks like

$$\mathbf{V}(\tau) \sim \int_0^{\tau} e^{-(\tau - s)r} \mathbf{G}_0(\tau, s)\beta(s)ds + \cdots$$

Potential next steps III

Liquidity risk

- Incorporate liquidity risk in the behaviour of the risky asset
- Leads to diffusion-degenerate nonlinear Thiele equation with error term quadratic in the spatial gradient of the reserve
- Solution given as weak solution in an L^2-space

Contents

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