How superadditive can a risk measure be?

Valeria Bignozzi[1]

joint work with Andreas Tsanakas[2] and Ruodu Wang[3]

[1]Dept of Mathematics, RiskLab, ETH Zurich
[2]Cass Business School, City University London
[3]Dept of Statistics and Actuarial Science, University of Waterloo

Risk Management and Risk Measures,
Hannover, 28 May 2014

Motivation

- What are the properties that a risk measure should satisfy?
  - Diversification & subadditivity (Artzner *et al.*, 1999; Dhaene *et al.*, 2008)
  - Estimation & robustness (Cont *et al.*, 2010; Krätschmer *et al.*, 2013)
  - Backtesting & elicitability (Gneiting, 2011; Ziegel, 2014; Bellini and B., 2013)
- Embrechts *et al.* (2014); Emmer *et al.* (2013)

Our contribution

Assume a risk measure is NOT subadditive, i.e. there exist losses X, Y such that

$$\rho(X+Y) > \rho(X) + \rho(Y),$$

HOW MUCH SUPERADDITIVE CAN IT BE?

- Aggregation of positions may be penalized
- Quantifying worst-case scenario
- No obvious upper bound for the risk of the aggregate position
- Measure model/dependence uncertainty

Our contribution

- For distortion risk measures:
  - The boundary is given by the smallest coherent distortion risk measure dominating the risk measure
- For shortfall risk measures:
  - The boundary is given by the smallest coherent expectile dominating the risk measure
- Further risk measures are considered in the paper

We lend support to coherent risk measures...

Risk measures

- ▷ Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space
- ▷ $L^0 = L^0(\Omega, \mathcal{F}, \mathbb{P})$ is the space of all measurable random variables
- ▷ $X \in L^0$ represents a financial loss
- ▷ When needed we denote X_F, a random variable $X \sim F$

A risk measure is any functional

$$\rho : \; L^0 \; \rightarrow \; \mathbb{R} \cup \{-\infty, +\infty\}$$

Standard properties for risk measures

The following properties are assumed throughout the presentation: for any $X, Y \in L^0$

- ▷ Law-invariance: If $X, Y \sim F$, then $\rho(X) = \rho(Y)$
- ▷ Monotonocity: If $X \geq Y$ then $\rho(X) \geq \rho(Y)$
- ▷ Cash-invariance: $\forall m \in \mathbb{R}$, $\rho(X - m) = \rho(X) - m$
- ▷ Normalization: $\rho(0) = 0$

Key extra properties

For this presentation we focus on risk measures that satisfy

- Convexity:
  $\forall \lambda \in [0,1],\ \rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$
  - Convex risk measures (Föllmer and Schied, 2002, Frittelli and Rosazza Giannin, 2002), and/or

Key extra properties

For this presentation we focus on risk measures that satisfy

- Convexity:
  $\forall\lambda \in [0,1]$, $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$
  - Convex risk measures (Föllmer and Schied, 2002, Frittelli and Rosazza Giannin, 2002), and/or
- Positive homogeneity: $\forall\alpha \geq 0$, $\rho(\alpha X) = \alpha\rho(X)$
  - Coherent risk measures (Artzner et al. 1999), and/or

Key extra properties

For this presentation we focus on risk measures that satisfy

- ▷ Convexity:
  $\forall \lambda \in [0,1]$, $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$
  - Convex risk measures (Föllmer and Schied, 2002, Frittelli and Rosazza Giannin, 2002), and/or
- ▷ Positive homogeneity: $\forall \alpha \geq 0$, $\rho(\alpha X) = \alpha\rho(X)$
  - Coherent risk measures (Artzner et al. 1999), and/or
- ▷ Comonotonicity: If X and Y are comonotonic,
  $\rho(X+Y) = \rho(X) + \rho(Y)$

Key extra properties

For this presentation we focus on risk measures that satisfy

- Convexity:
  $\forall \lambda \in [0,1]$, $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$
  - Convex risk measures (Föllmer and Schied, 2002, Frittelli and Rosazza Giannin, 2002), and/or

- Positive homogeneity: $\forall \alpha \geq 0$, $\rho(\alpha X) = \alpha\rho(X)$
  - Coherent risk measures (Artzner et al. 1999), and/or

- Comonotonicity: If X and Y are comonotonic, $\rho(X+Y) = \rho(X) + \rho(Y)$

Subadditivity (NOT ASSUMED):

$$\rho(X+Y) \leq \rho(X) + \rho(Y)$$

Classical examples

- Value-at-Risk is positively homogeneous and comonotonic but not subadditive

$$\mathrm{VaR}_p(X) = \inf\{x : \mathbb{P}(X \le x) \ge p\}, \qquad p \in (0,1)$$

Classical examples

- Value-at-Risk is positively homogeneous and comonotonic but not subadditive

$$\mathrm{VaR}_p(X) = \inf\{x : \mathbb{P}(X \leq x) \geq p\}, \qquad p \in (0,1)$$

- Expected Shortfall is positively homogeneous, comonotonic and subadditive

$$\mathrm{ES}_p(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_\alpha(X)\mathrm{d}\alpha, \qquad p \in [0,1)$$

Classical examples

- Value-at-Risk is positively homogeneous and comonotonic but not subadditive

$$\mathrm{VaR}_p(X) = \inf\{x : \mathbb{P}(X \leq x) \geq p\}, \qquad p \in (0,1)$$

- Expected Shortfall is positively homogeneous, comonotonic and subadditive

$$\mathrm{ES}_p(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_\alpha(X)\mathrm{d}\alpha, \qquad p \in [0,1)$$

- Entropic risk measure is convex, but not positively homogeneous, comonotonic or subadditive

$$\mathrm{ER}_\lambda(X) = \frac{1}{\lambda}\log \mathbb{E}[e^{\lambda X}], \qquad \lambda > 0$$

The lack of subadditivity...

For comonotonic (or positive homogeneous) risk measures implies:

- $X, Y \in L^0$, $X, Y \sim F$ such that

$$\rho(X+Y) > \rho(X) + \rho(Y)$$

but...

- For $X^c, Y^c \sim F$ comonotonic

$$\rho(X^c + Y^c) = \rho(X^c) + \rho(Y^c) < \rho(X+Y)$$

- Comonotonic risks do not represent the worst-case dependence
- Inconsistent ordering of risk (Bäuerle and Müller, 2006)
- VaR

The lack of subadditivity...

For convex, normalized (not homogeneous) risk measures implies:

- For X, Y, X^c, Y^c as before

$$\rho(X^c + Y^c) \geq \rho(X + Y) > \rho(X) + \rho(Y)$$

- Comonotonic risks represent the worst-case dependence
- There is no inconsistent ordering, but an aggregation penalty designed in the risk measure
- E.g. for including liquidity risks

$$\rho(nX) \geq n\rho(X) \qquad n \geq 1$$

- Entropic Risk Measure

Superadditivity ratio

- Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a vector of risks
- $X_i \in L^0$, $X_i \sim F_i$ for $i = 1, \ldots, n$ (for now)
- $\rho(X_i) \in (0, \infty)$ for $i = 1, \ldots, n$ (for now)

Superadditivity ratio

$$\Delta^{\mathbf{X}}(\rho) = \frac{\rho(X_1 + \ldots + X_n)}{\rho(X_1) + \ldots + \rho(X_n)}, \qquad n \in \mathbb{N}$$

- $\Delta^{\mathbf{X}}(\rho) \leq 1$ for subadditive risk measures

Superadditivity ratio (Cont'ed)

- For a homogeneous portfolio:

$$\Delta_n^F(\rho) = \sup\left\{ \frac{\rho(X_1 + \ldots + X_n)}{\rho(X_1) + \ldots + \rho(X_n)},\ X_1, \ldots, X_n \sim F \right\}$$

- Law-invariant
- $\rho(X_F) \in (0, \infty)$
- Worst-case superadditivity for a given portfolio size n
- Worst-case dependence structure (Bernard et al., 2014)

Superadditivity ratio (Cont'ed)

- $\mathfrak{S}_n(F) := \{X_1 + \ldots + X_n : X_i \sim F, i = 1, \ldots, n\}$

$$\Delta_n^F(\rho) = \frac{1}{n\rho(X_F)} \sup\{\rho(S) : S \in \mathfrak{S}_n(F)\}$$

- The hypothesis $\rho(X_F) \in (0, \infty)$ is not mathematically required

- We define

$$\Gamma_{\rho,n}(X_F) = \frac{1}{n} \sup\{\rho(S) : S \in \mathfrak{S}_n(F)\}$$

The extreme aggregation measure

The extreme aggregation measure (Slightly cheating!)

$$\Gamma_\rho(X_F) = \sup_{n\in\mathbb{N}}\{\frac{1}{n}\sup\{\rho(S) : S \in \mathfrak{S}_n(F)\}\}$$
$$= \lim_{n\to\infty}\{\frac{1}{n}\sup\{\rho(S) : S \in \mathfrak{S}_n(F)\}\}$$

- ρ is comonotonic and/or positive homogeneous and/or convex with $\rho(0) = 0$
- If well defined

$$\sup_{n\in\mathbb{N}} \Delta_n^F(\rho) = \sup_{n\in\mathbb{N}}\left\{\frac{1}{n\rho(X_F)}\sup\{\rho(S) : S \in \mathfrak{S}_n(F)\}\right\} = \frac{\Gamma_\rho(X_F)}{\rho(X_F)}$$

The extreme aggregation measure

$$\Gamma_\rho : L^0 \to [-\infty, +\infty]$$

Lemma

- Γ_ρ is a law-invariant risk measure
- It inherits the properties of monotonicity, cash-invariance, positive homogeneity, subadditivity, convexity, normality, from ρ
- Given any subadditive risk measure ρ^+ dominating ρ, $\Gamma_\rho \leq \rho^+$
- Generally $\Gamma_\rho \geq \rho$

Superadditivity of distortion risk measures

- Assume here that random variables are bounded from below, i.e. $F^{-1}(0) > -\infty$
- A distortion risk measure is defined as

$$\rho_h(X_F) = \int_0^1 \mathrm{VaR}_\alpha(X_F)\mathrm{d}h(\alpha)$$

where

  - h is an increasing, right-continuous and left-limit function, with $h(0) = h(0+) = 0$ and $h(1-) = h(1) = 1$ (Wang et al.,1997)

Properties of distortion risk measures

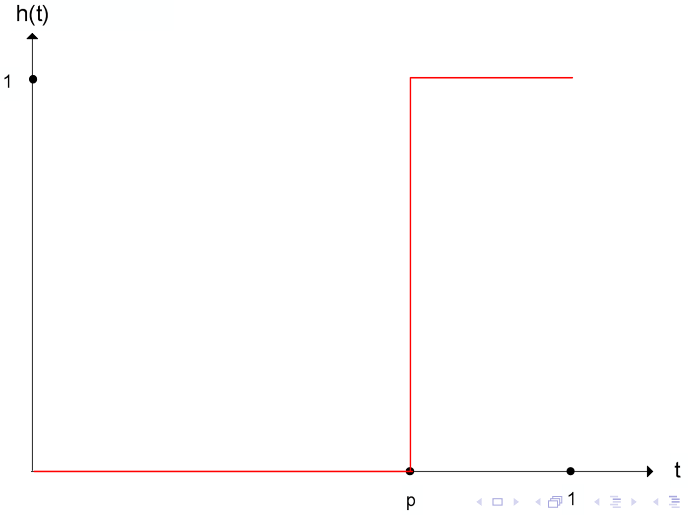
- Law-invariant, monotone, cash-invariant, positively homogeneous, comonotonic
- If h is convex then ρ_h is coherent (Acerbi, 2002)
- VaR, ES

Properties of distortion risk measures

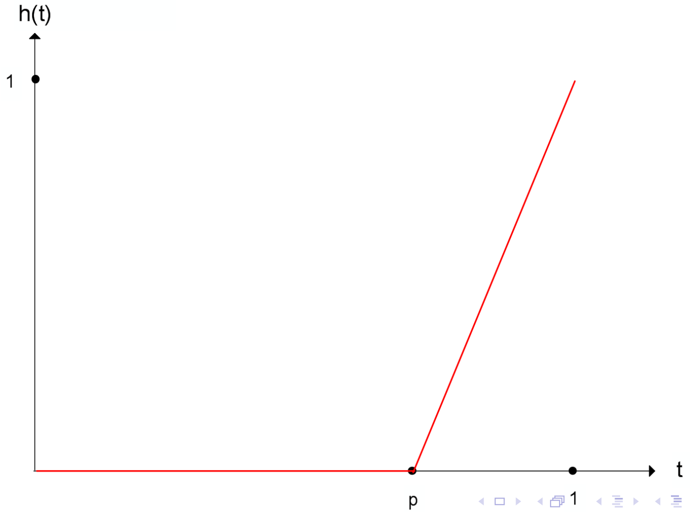
- Law-invariant, monotone, cash-invariant, positively homogeneous, comonotonic
- If h is convex then ρ_h is coherent (Acerbi, 2002)
- VaR, ES
- A robust alternative to expected shortfall is Range-Value-at-Risk (Cont *et al.*, 2010):

$$\mathrm{RVaR}_{p,q}(X_F) = \frac{1}{q-p}\int_p^q VaR_\alpha(X_F)\mathrm{d}\alpha \quad 0 \leq p < q < 1$$

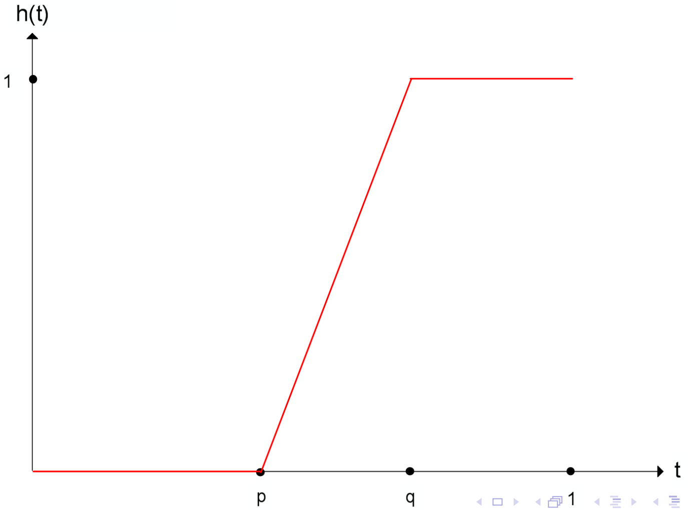
Distortion function of VaR_p



Distortion function of ES_p



Distortion function of $\text{RVaR}_{p,q}$



How superadditive can a distortion risk measure be?

For the risk measures VaR_p and $\mathrm{RVaR}_{p,q}$:

$$\sup_{n\in\mathbb{N}} \Delta_n^F(VaR_p) = \frac{\Gamma_{\mathrm{VaR}_p}(X_F)}{VaR_p(X_F)} = \frac{\mathrm{ES}_p(X_F)}{\mathrm{VaR}_p(X_F)}$$

- Puccetti and Rüschendorf (2014), complete mixability
- Puccetti *et al.* (2013), strictly positive densities
- Wang (2014), bounded densities
- Wang and Wang (2014), for any distribution
- The same holds for RVaR_p

Main Theorem

- Let ρ_h^+ be the smallest coherent distortion risk measure dominating ρ_h

Lemma

ρ_h^+ exists and is given by $\rho_h^+ = \rho_{h^*}$, where for $t \in [0,1]$,

$$h^*(t) = \sup\{g(t) : g : [0,1] \to [0,1],\ g \leq h,\ g \text{ is increasing, and convex on } [0,1]\}$$

Main Theorem

Main Theorem

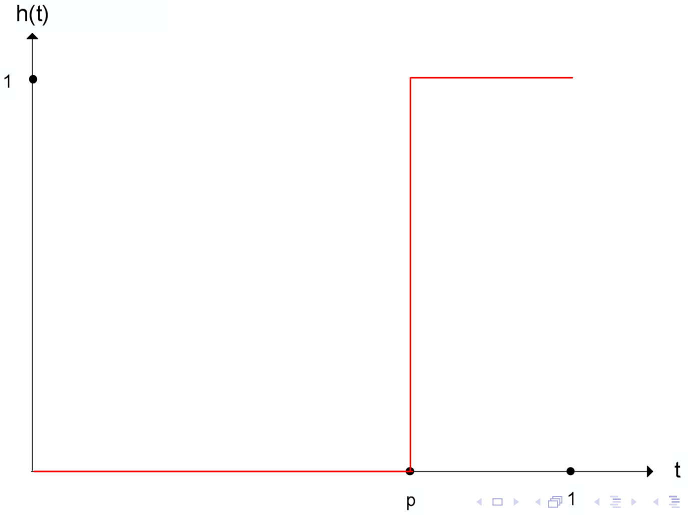
For any distortion risk measure ρ_h, the extreme aggregation measure is $\Gamma_{\rho_h} = \rho_h^+$ (10 pages proof...)

Corollary

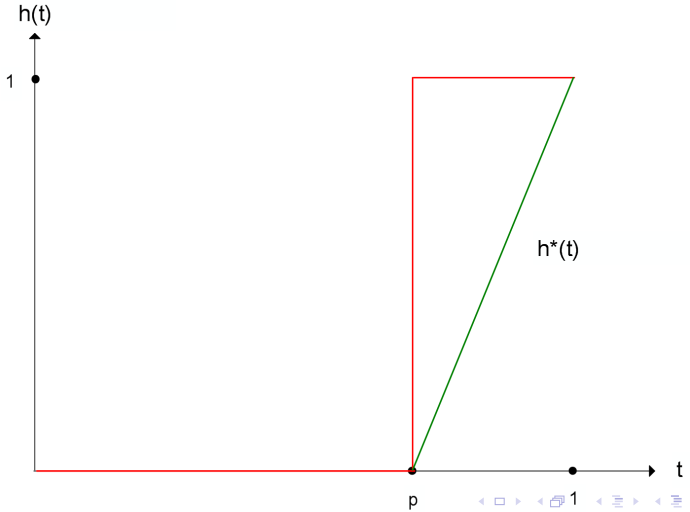
Γ_{ρ_h} is the smallest coherent risk measure dominating ρ

- Γ_{ρ_h} inherits all the properties of ρ_h (including comonotonicity)
- Γ_{ρ_h} gains subadditivity (coherency)

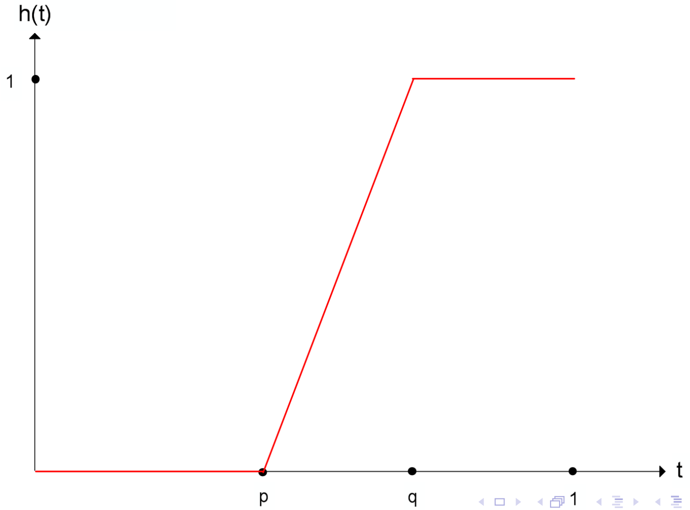
VaR$_p$: functions h and h^*



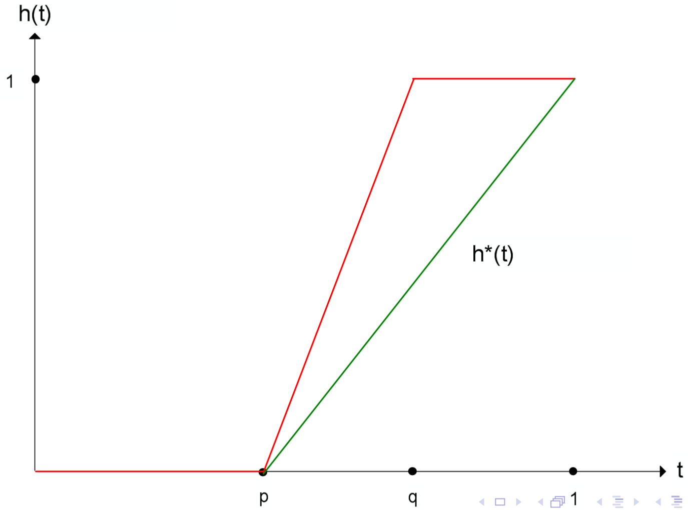
VaR_p: functions h and h^*



$\mathrm{RVaR}_{p,q}$: functions h and h^*



$\mathrm{RVaR}_{p,q}$: functions h and h^*



For general distortion risk measures...

- Assume that h is piecewise linear
- Assume that F has bounded support
- Add 10 pages of proof!

We generalize this result to

$$\rho_G(X_F) = \sup_{h \in G} \rho_h(X_F) \qquad \Gamma_{\rho_G} = \sup_{h \in G} \rho_h^+(X_F)$$

Convex shortfall risk measures

A convex shortfall risk measure ρ_ℓ is defined as the unique solution of

$$\mathbb{E}[\ell(X - x)] = 0,$$

where ℓ is an increasing not identically constant convex function with 0 in the interior of its range

Convex shortfall risk measures

A convex shortfall risk measure ρ_ℓ is defined as the unique solution of

$$\mathbb{E}[\ell(X - x)] = 0,$$

where ℓ is an increasing not identically constant convex function with 0 in the interior of its range

- Assume all X in L^1
- Convex measures of risk (Föllmer and Schied, 2011)
- ER measure generated by $\ell(x) = \exp(\lambda x) - 1$

Loss functions

- Since ℓ is convex, we know that

$$a_\ell := \lim_{x \to \infty} \ell'(x) \text{ exists in } [0, \infty]$$

$$b_\ell := \lim_{x \to -\infty} \ell'(x) \text{ exists in } [0, \infty)$$

$$b_\ell \leq a_\ell$$

Loss functions

- Since ℓ is convex, we know that

$$a_\ell := \lim_{x \to \infty} \ell'(x) \text{ exists in } [0, \infty]$$
$$b_\ell := \lim_{x \to -\infty} \ell'(x) \text{ exists in } [0, \infty)$$
$$b_\ell \leq a_\ell$$

- Define the convex loss function

$$\ell^*(x) = a_\ell x_+ - b_\ell x_-$$

Loss functions

- Since ℓ is convex, we know that

$$a_\ell := \lim_{x\to\infty} \ell'(x) \text{ exists in } [0,\infty]$$
$$b_\ell := \lim_{x\to-\infty} \ell'(x) \text{ exists in } [0,\infty)$$
$$b_\ell \leq a_\ell$$

- Define the convex loss function
$\ell^*(x) = a_\ell x_+ - b_\ell x_-$
- The shortfall risk measure ρ_{ℓ^*} is identified with e_{p_ℓ}, the p_ℓ-expectile (Newey and Powell, 1987), for
$p_\ell = \frac{a_\ell}{a_\ell + b_\ell}$
- Expectiles are defined as the unique solution to

$$p\mathbb{E}[(X-x)_+] - (1-p)\mathbb{E}[(X-x)_-] = 0, \quad p \in (0,1)$$

- We define

$$e_0(X_F) = \text{ess-inf} X_F, \qquad e_1(X_F) = \text{ess-sup}(X_F)$$

Extreme-scenario measures

Theorem

For any shortfall risk measure ρ_ℓ, *it is* $\Gamma_{\rho_\ell} = \rho_{\ell^*} = e_{p_\ell}$.

Extreme-scenario measures

Theorem

For any shortfall risk measure ρ_ℓ, it is $\Gamma_{\rho_\ell} = \rho_{\ell^*} = e_{p_\ell}$.

Corollary

Γ_{ρ_ℓ} is the smallest coherent risk measure dominating ρ_ℓ

- Once more Γ_{ρ_ℓ} is coherent, even though ρ_ℓ is not!
- Expectiles make an appearance as the only elicitable coherent shortfall risk measures (Ziegel, 2013, Bellini and B. 2013).

The Entropic Risk Measure

- The $\mathrm{ER}_1(X) = \log(\mathbb{E}(e^X))$ is consistent with second order stochastic dominance
- Worst-case dependence is given by comonotonic risks

$$\begin{aligned}\Gamma_{\mathrm{ER}_1,n}(X_F) &= \frac{1}{n}\sup\{\mathrm{ER}_1(S) : S \in \mathfrak{S}_n(F)\} \\ &= \text{ess-sup}(X_F) \\ &= e_1(X_F)\end{aligned}$$

Conclusion

- Γ_ρ gains positive homogeneity, subadditivity and convexity in all cases studied
- It is always a coherent risk measure
- We do not have yet a universal result on this

Even if you work with a non-coherent risk measure, under dependence uncertainty its extreme behavior leads to coherency...

Thank you for your kind attention!

- F. Bellini and V. Bignozzi *Elicitable risk measures.* Preprint ETH
- R. Cont, R. Deguest, G. Scandolo *Robustness and sensitivity analysis of risk measurement procedures* Quantitative Finance, Vol 10, No. 6, 2010
- P. Embrechts, G. Puccetti, L. Rüschendorf, R. Wang, A. Beleraj *An Academic response to Basel 3.5.* Risks, Vol 2, No. 1, 2014.
- S. Emmer, M. Kratz, D. Tasche *What is the best risk measure in practice?A comparison of standard measures.* Preprint, ESSEC Business School, 2014. Journal of the American Statistical Association, Vol 106, No. 494, 2011.
- J. Ziegel *Coherence and elicitability.* To appear in Mathematical Finance, 2014.