On the dual of the solvency cone

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The setting

- d assets with bid-ask prices, modeled by solvency cone K_d (fixed time t and state ω)

- Varying t: cone-valued stochastic process $(K_t)_{t=0}^T$

  (replaces stock price process $(S_t)_{t=0}^T$ in frictionless market)

- Consistent price process is a martingale $(Z_t)_{t=0}^T$ with $Z_t \in K_t^+$ (positive dual cone) $P-a.s.$ for all t

  (replaces equivalent martingale measures in frictionless market)

The question

Generating vectors of K_d^+? (Calculation? How many? Is there a structure?)

Posed as an open problem in Bouchard, Touzi (2000, AAP)

Why important?

Characterize efficient trades:

- A portfolio $x \in \mathbb{R}^d$ can be traded into $y \in x - K_d$
- but only trades on the boundary of $x - K_d$ (i.e. the faces of $x - K_d$) are reasonable (do not burn money)
- **faces of** K_d correspond to generating vectors of K_d^+

Dual variables:

Play the role of equivalent martingale measures: appear in dual characterization of superhedging, portfolio optimization, market-risk measures, ... in markets with proportional transaction cost (and even in limit order book markets)

Algorithm:

K_d^+ needed as an input in algorithms to compute superhedging prices, market-risk measures in transaction cost markets

The results (Löhne, Rudloff (2014), Forthcoming at Discrete Applied Mathematics.)

- Complete characterization of K_d^+ (structure, upper and lower bound for number, exact number for important special cases) for **arbitrary dimension** d

- **Algorithm** to compute K_d^+

- For special cases no algorithm necessary as K_d^+ has a **simple recursive structure**

- Uses **graph theory, combinatorial optimization**

The starting point

- easy for $d = 2$ and $d = 3$
- no clue for $d \geq 4$...
- brutal force gives generating vectors of dual in numerical examples (until $d = 7$) by vertex enumeration (very expensive)
- no structural results ...

$\lvert K_d \rvert,\ \lvert K_d^+ \rvert$	$d = 2$	3	4	5	6	7	...	d
general	2, 2	6, 6	12, 20	20, 70	30, 252	42, 924	...	$d(d-1)$, ???
case 1	2, 2	6, 6	12, 14	20, 30	30 , 62	42, 126	...	$d(d-1)$, $2^d - 2$?
case 2	2, 2	4, 4	6, 8	8 , 16	10, 32	12, 64	...	$2(d-1)$, 2^{d-1}?

case 1: d currencies with positive bid-ask-spread.
case 2: d assets all denoted in domestic currency (= asset 1), exchanges only via domestic currency.

The final result

$\lvert K_d \rvert, \lvert K_d^+ \rvert$	$d = 2$	3	4	...	7	...	d
general	2, 2	6, 6	12, 20	...	42, 924	...	$d(d-1)$, $\sum_{p=1}^{d-1} \binom{d-2}{p-1} \binom{d}{p}$
case 1	2, 2	6, 6	12, 14	...	42, 126	...	$d(d-1)$, $2^d - 2$
case 2	2, 2	4, 4	6, 8	...	12, 64	...	$2(d-1)$, 2^{d-1}

case 1: d currencies with positive bid-ask-spread.
case 2: d assets all denoted in domestic currency (= asset 1), exchanges only via domestic currency.

The final result

Recursive representation in special cases:

E.g. case 2: bid and ask prices $b_i < a_i$ for $i \in \{2, ..., d\}$ expressed by asset 1 ('cash').
For $d \geq 3$ (columns of Y_d are generating vectors of K_d^+)

$$Y_2 = \begin{pmatrix} 1 & 1 \\ a_2 & b_2 \end{pmatrix} \qquad Y_d = \begin{pmatrix} & Y_{d-1} & & & Y_{d-1} & \\ & & & & & \\ a_d & \dots & a_d & b_d & \dots & b_d \end{pmatrix}.$$

The details

Definition (solvency cone)

π_{ij} : number of units of asset i for which an agent can buy one unit of asset j.

Let $d \in \{2, 3, \ldots\}$, $V = \{1, \ldots, d\}$ and let $\Pi = (\pi_{ij})$ be a $(d \times d)$-matrix such that

$$\forall i \in V: \quad \pi_{ii} = 1, \tag{1}$$

$$\forall i, j \in V: \quad 0 < \pi_{ij}, \tag{2}$$

$$\forall i, j, k \in V: \quad \pi_{ij} \leq \pi_{ik}\pi_{kj}, \tag{3}$$

$$\exists i, j, k \in V: \quad \pi_{ij} < \pi_{ik}\pi_{kj}. \tag{4}$$

Sometimes, (3) and (4) is replaced by (efficient frictions)

$$\forall i, j \in V,\ \forall k \in V \setminus \{i, j\}: \quad \pi_{ij} < \pi_{ik}\pi_{kj}. \tag{5}$$

The polyhedral convex cone

$$K_d := \text{cone}\left\{\pi_{ij}e^i - e^j \middle|\ ij \in V \times V\right\}$$

is called solvency cone induced by Π.

The dual cone

$K_d^+ := \left\{ y \in \mathbb{R}^d | \; \forall x \in K_d : \; x^T y \geq 0 \right\}$ ... (positive) dual cone of K_d

Trivial: generating vectors of solvency cone give inequality representation of dual cone:

Proposition 1. One has $K_d^+ = \left\{ y \in \mathbb{R}^d | \; \forall i, j \in V : \pi_{ij} y_i \geq y_j \right\}$.

Proof: obvious
Recall: $K_d :=$ cone $\left\{ \pi_{ij} e^i - e^j | \; ij \in V \times V \right\}$

Thus, vertex enumeration gives generating vectors of dual in numerical examples.

Generating vectors of dual cone correspond to faces of the primal cone (efficient trades!)

Bi-partitions

$V = \{1, \ldots, d\}$
(P, N) … bi-partition of V, i.e., $\emptyset \neq P \subsetneq V$, $N = V \setminus P$

Motivation for use of bi-partitions:

Cone K_d has faces in any orthant in $\mathbb{R}^d$ (except in $\mathbb{R}^d_+$ and $\mathbb{R}^d_-$). All points in one of those orthants correspond to a bi-partition: let $x \in \mathbb{R}^d$. Collect $i \in P$ (Positive) if $x_i > 0$ and $j \in N$ (Negative) if $x_j \leq 0$.

Want to find all faces of K_d in a given orthant (= a given bi-partition).

Feasible tree solution

$V = \{1, \ldots, d\}$
(P, N) ... bi-partition of V, i.e., $\emptyset \neq P \subsetneq V$, $N = V \setminus P$
$G(P, N)$ ... bi-partite digraph with arc set $E = P \times N$

Spanning tree of $G(P, N)$ ... connected, no cycles ($d - 1$ edges)

$y \in \mathbb{R}^d$ is called feasible tree solution w.r.t (P, N) if there is a spanning tree T of $G(P, N)$ such that

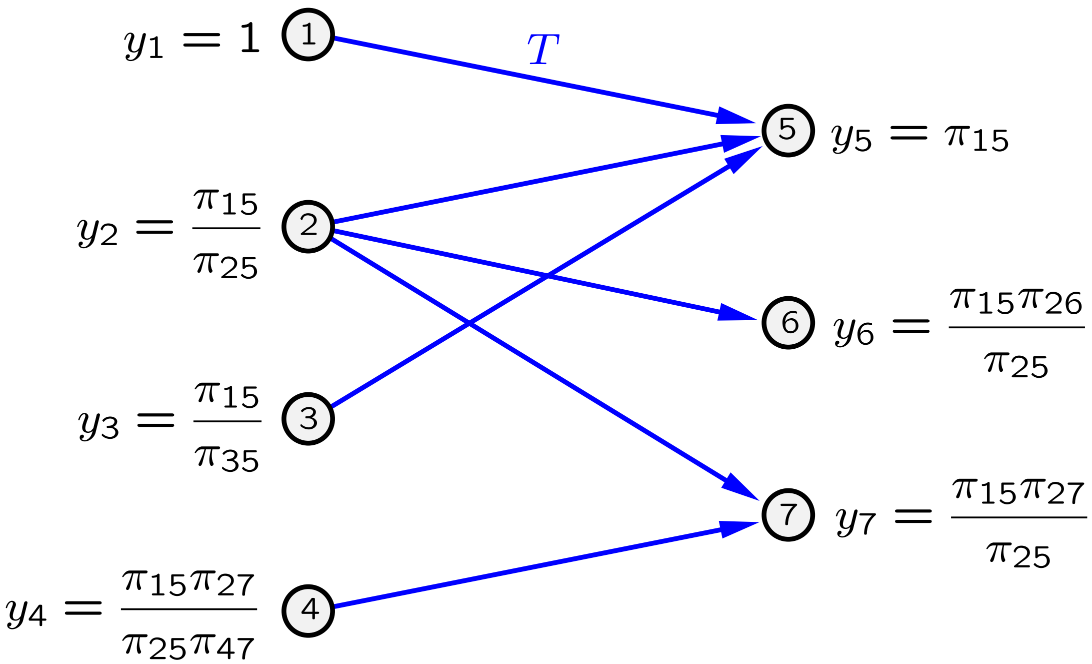
$$\forall ij \in E(T) \subseteq P \times N : \ \pi_{ij} y_i = y_j > 0. \tag{6}$$

and

$$\forall ij \in P \times N : \pi_{ij} y_i \geq y_j > 0. \tag{7}$$

Feasible tree solution

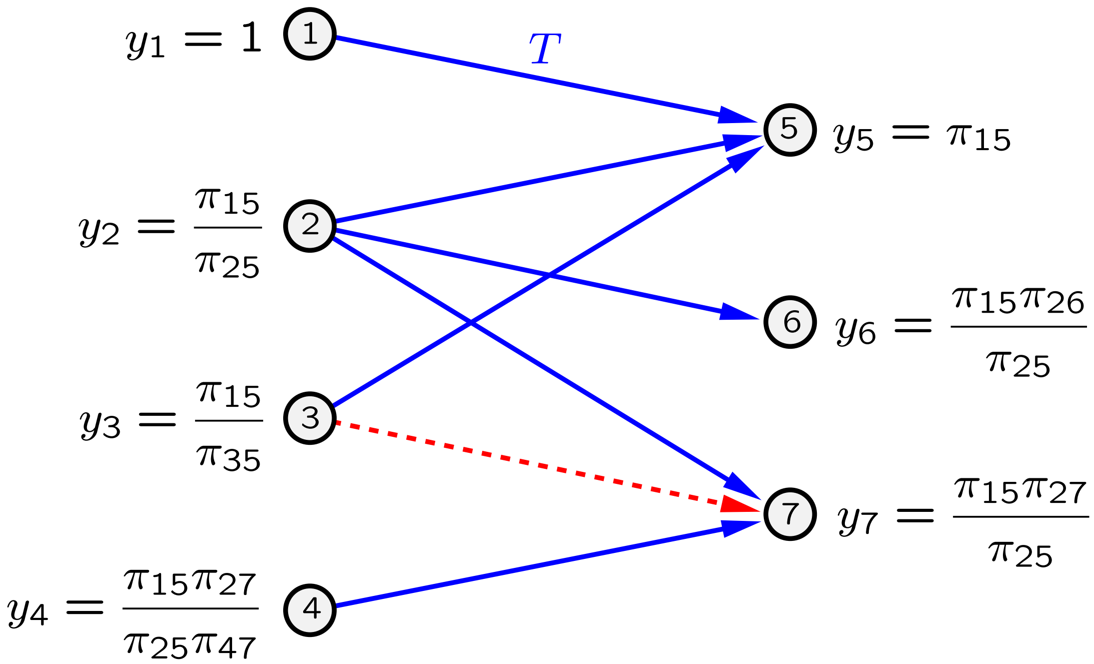
$V = \{1,2,3,4,5,6,7\},\ P = \{1,2,3,4\},\ N = \{5,6,7\}$



Tree solution: $\pi_{ij} y_i = y_j$ for $ij \in E(T)$

Feasible tree solution

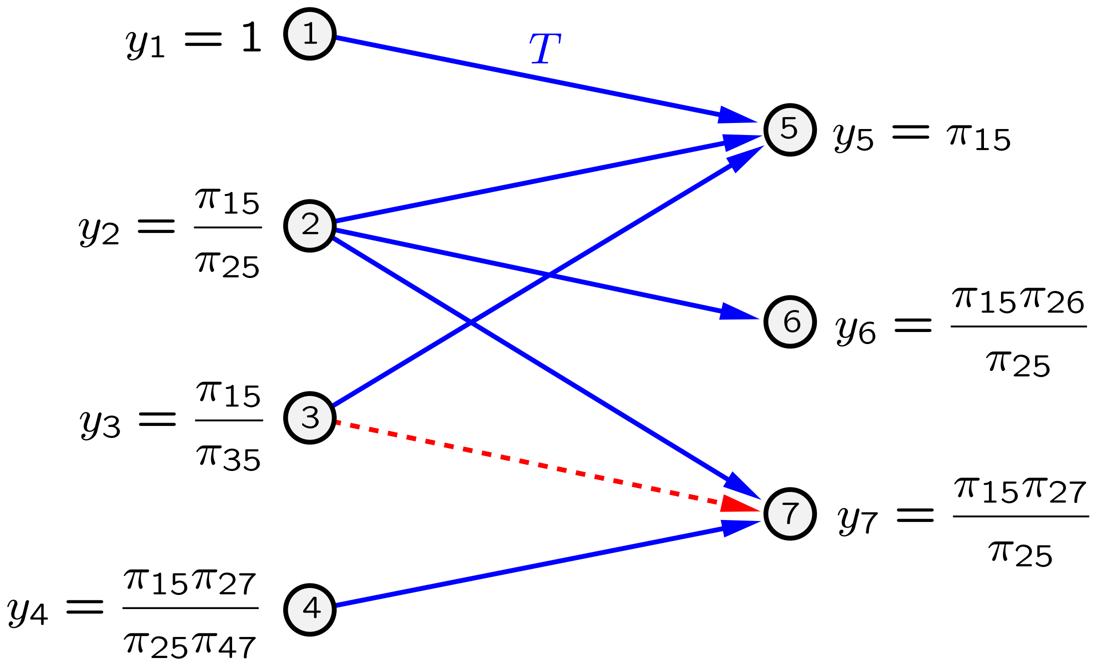
$$V = \{1,2,3,4,5,6,7\},\ P = \{1,2,3,4\},\ N = \{5,6,7\}$$



Feasibility: e.g. $\pi_{37} y_3 \geq y_7$

Feasible tree solution

$V = \{1, 2, 3, 4, 5, 6, 7\},\ P = \{1, 2, 3, 4\},\ N = \{5, 6, 7\}$



Feasibility: e.g. $\pi_{37} y_3 \geq y_7$ i.e. $\frac{\pi_{37}}{\pi_{35}} \geq \frac{\pi_{27}}{\pi_{25}}$

Characterization of K_d^+

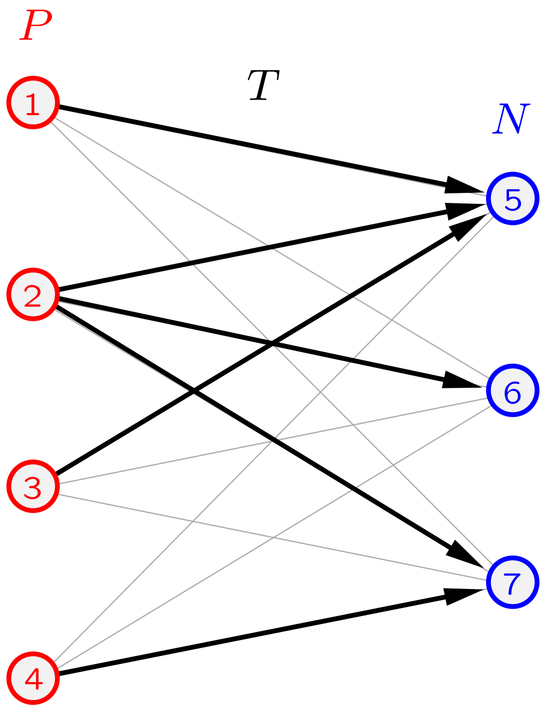
Theorem 1. For $y \in \mathbb{R}^d$, the following statements are equivalent.

(i) y is an extreme direction of K_d^+;

(ii) y is a feasible tree solution w.r.t. some bipartition (P, N) of V.

Degree vectors

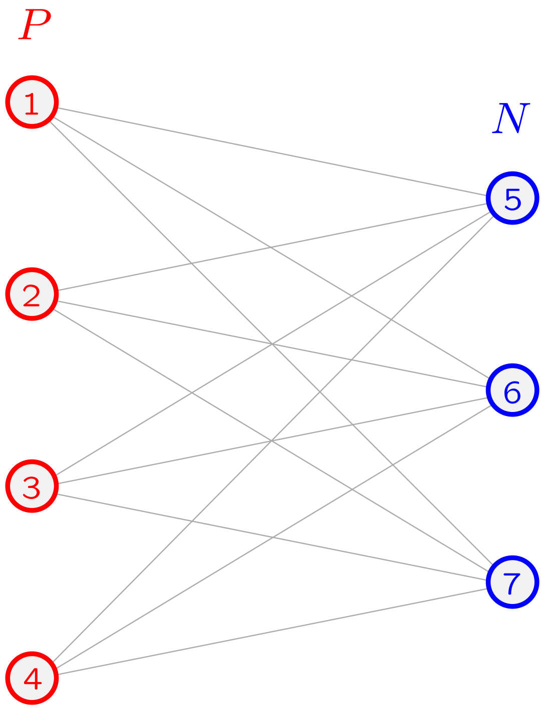
$$\deg_T(P) = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix}$$



$$\deg_T(N) = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

Degree vectors of spanning trees

$$\begin{pmatrix} 3 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 3 & 1 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 3 & 1 & 1 & 2 & 1 & 2 & 2 \end{pmatrix}$$



$$\begin{pmatrix} 4 & 1 & 1 & 3 & 3 & 2 & 1 & 2 & 1 & 2 \\ 1 & 4 & 1 & 2 & 1 & 3 & 3 & 1 & 2 & 2 \\ 1 & 1 & 4 & 1 & 2 & 1 & 2 & 3 & 3 & 2 \end{pmatrix}$$

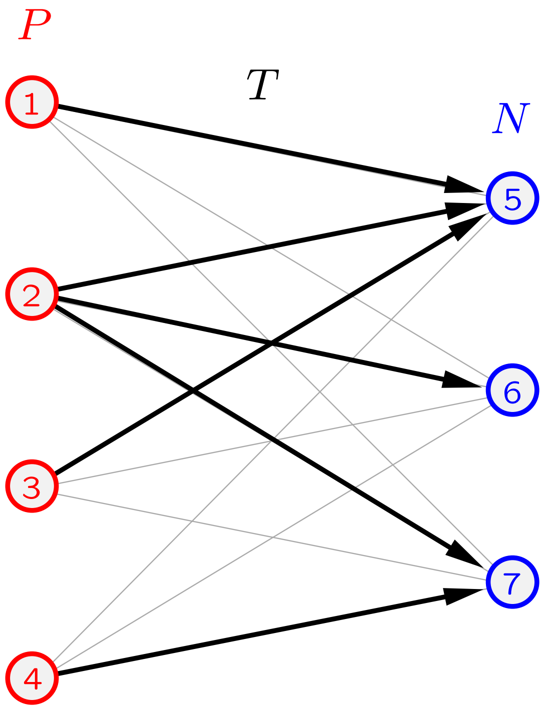
$c \in \mathbb{N}^P$ is called P-configuration if $\sum_{i \in P} c_i = d - 1$

$b \in \mathbb{N}^N$ is called N-configuration if $\sum_{i \in N} b_i = d - 1$

$$\mathbb{N} = \{1, 2, \ldots\}$$

Degree vectors of spanning trees

$$\begin{pmatrix} 3 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 3 & 1 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 3 & 1 & 1 & 2 & 1 & 2 & 2 \end{pmatrix}$$



$$\begin{pmatrix} 4 & 1 & 1 & 3 & 3 & 2 & 1 & 2 & 1 & 2 \\ 1 & 4 & 1 & 2 & 1 & 3 & 3 & 1 & 2 & 2 \\ 1 & 1 & 4 & 1 & 2 & 1 & 2 & 3 & 3 & 2 \end{pmatrix}$$

$c \in \mathbb{N}^P$ is called P-configuration if $\sum_{i \in P} c_i = d - 1$

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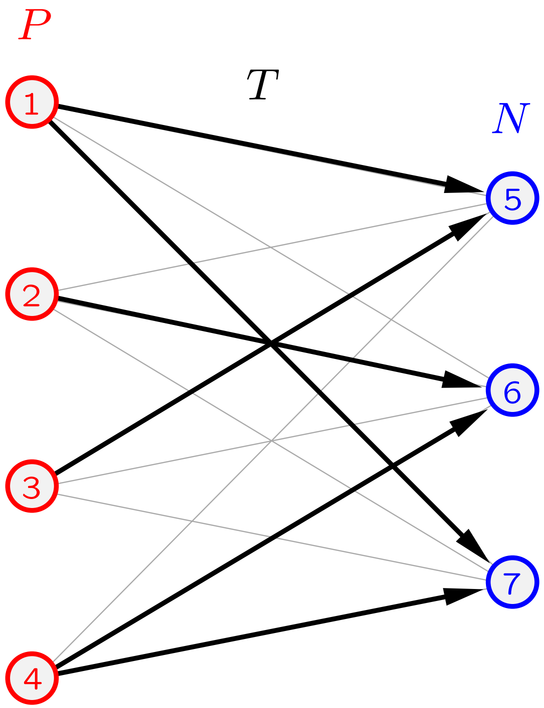
Degree vectors of spanning trees

$$\begin{pmatrix} 3 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 3 & 1 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 3 & 1 & 1 & 2 & 1 & 2 & 2 \end{pmatrix}$$



$$\begin{pmatrix} 4 & 1 & 1 & 3 & 3 & 2 & 1 & 2 & 1 & 2 \\ 1 & 4 & 1 & 2 & 1 & 3 & 3 & 1 & 2 & 2 \\ 1 & 1 & 4 & 1 & 2 & 1 & 2 & 3 & 3 & 2 \end{pmatrix}$$

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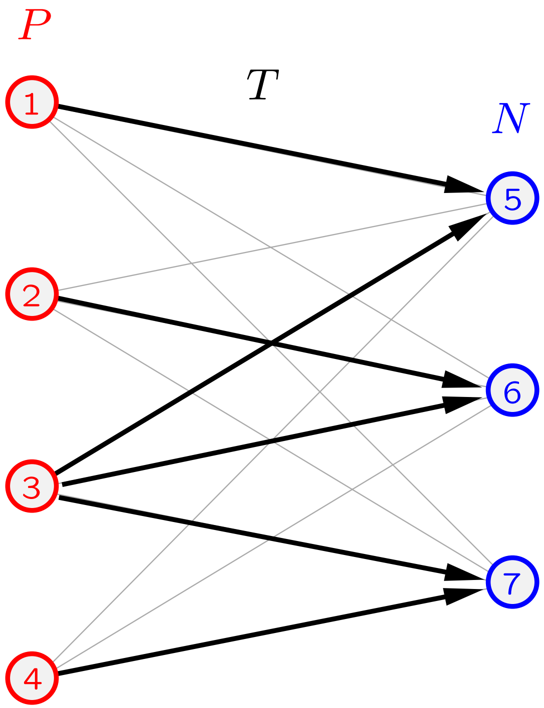
$\mathbb{N} = \{1, 2, \ldots\}$

Degree vectors of spanning trees

$$\begin{pmatrix} 3 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 3 & 1 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 3 & 1 & 1 & 2 & 1 & 2 & 2 \end{pmatrix}$$



$$\begin{pmatrix} 4 & 1 & 1 & 3 & 3 & 2 & 1 & 2 & 1 & 2 \\ 1 & 4 & 1 & 2 & 1 & 3 & 3 & 1 & 2 & 2 \\ 1 & 1 & 4 & 1 & 2 & 1 & 2 & 3 & 3 & 2 \end{pmatrix}$$

$c \in \mathbb{N}^P$ is called P-configuration if $\sum_{i \in P} c_i = d - 1$

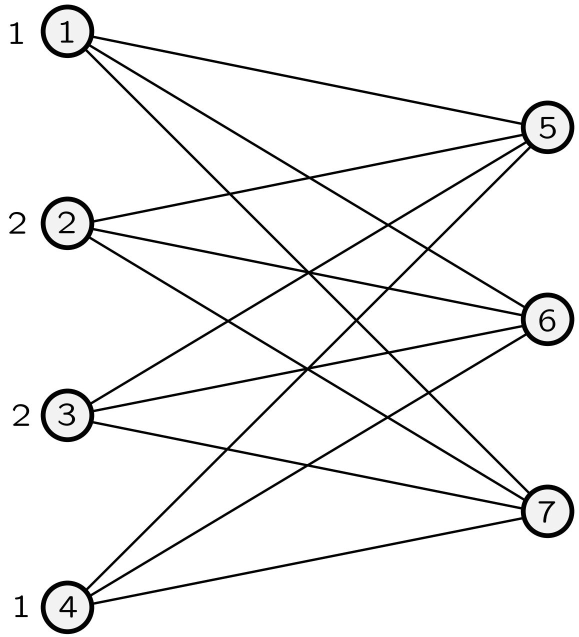
$b \in \mathbb{N}^N$ is called N-configuration if $\sum_{i \in N} b_i = d - 1$

$$\mathbb{N} = \{1, 2, \ldots\}$$

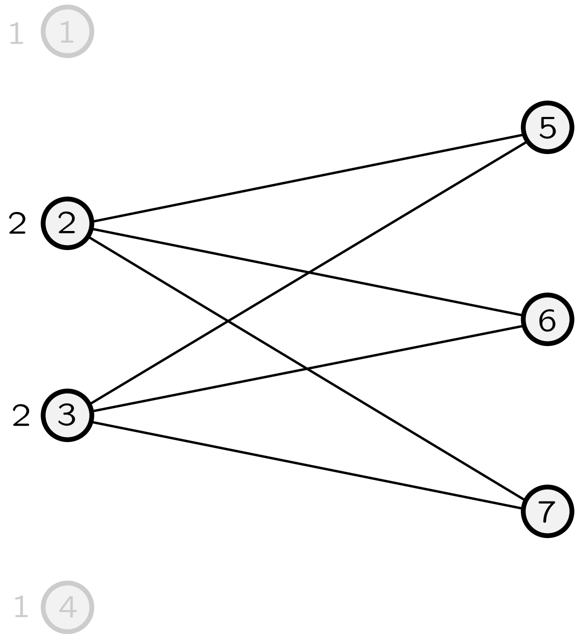
Existence of feasible tree solutions

Theorem 2. For every bi-partition (P, N) of V and every P-configuration $c \in \mathbb{N}^P$ there exists a feasible tree solution $y \in \mathbb{R}^d$ generated by a spanning tree T of the bi-partite graph $G(P, N)$ with $\deg_T(P) = c$. An analogous statement holds if an N-configuration is given.

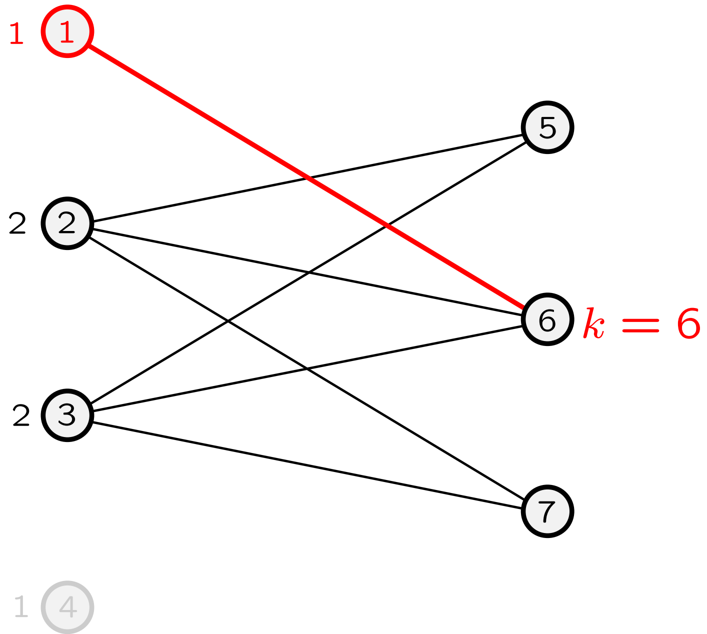
Towards a proof of Theorem 2



Towards a proof of Theorem 2

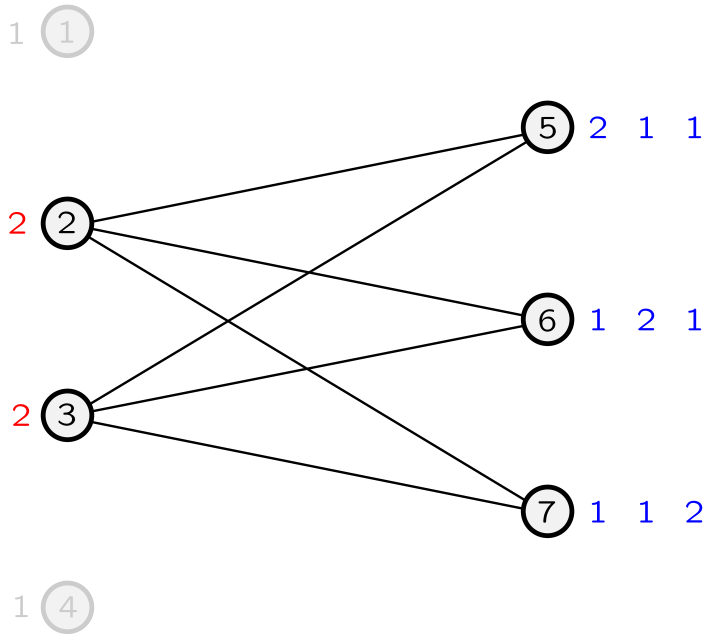


Towards a proof of Theorem 2



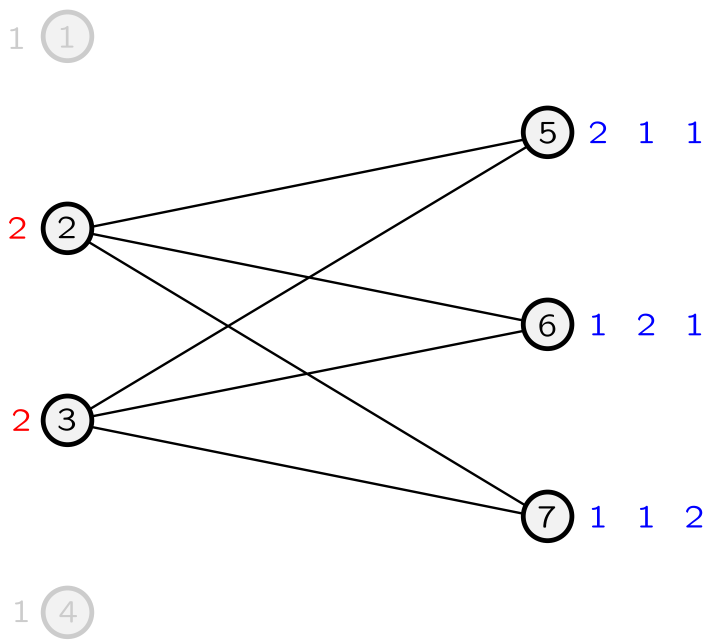
$$k \in \arg\max\{y_j/\pi_{1j} \mid j \in N\}$$

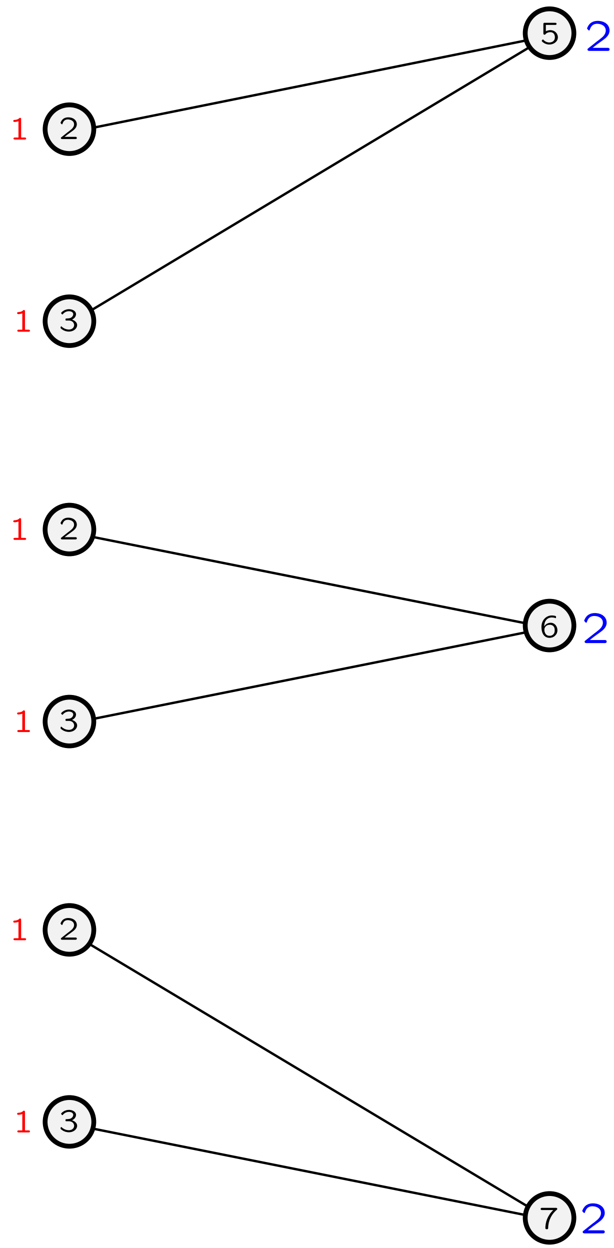
Towards a proof of Theorem 2



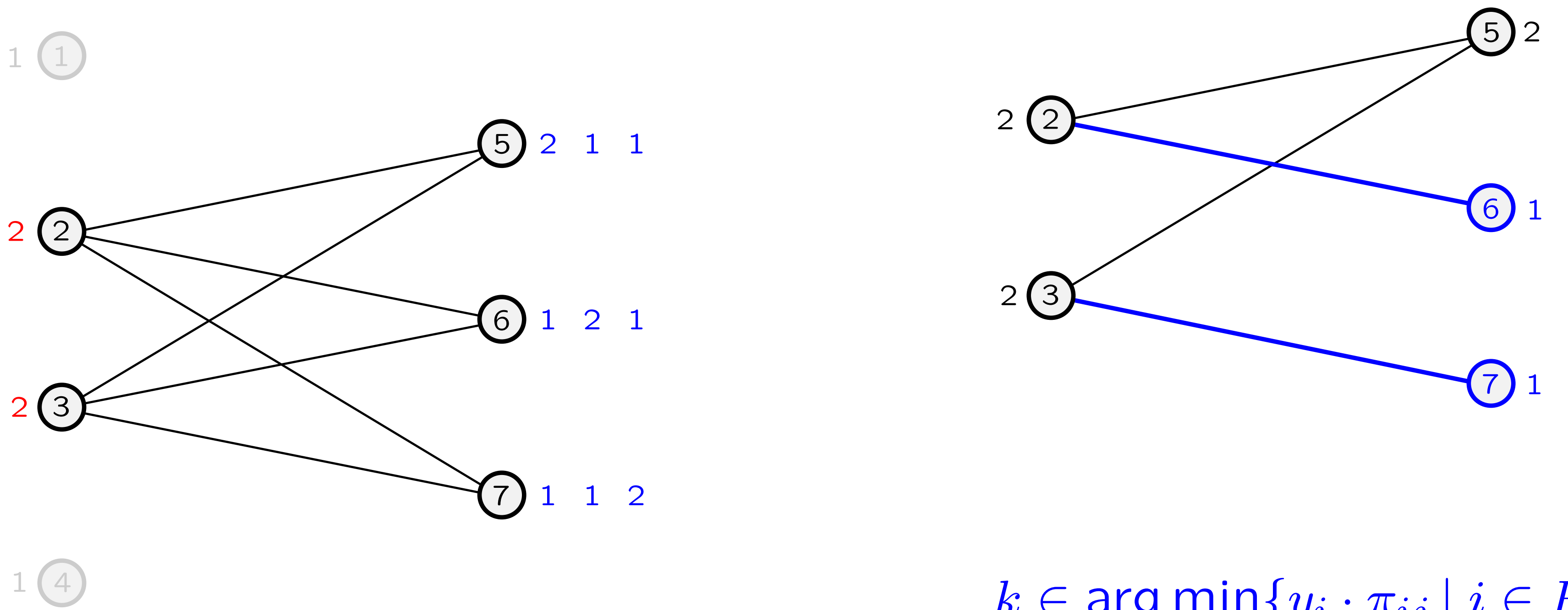
Is there an N-configuration $b \in \mathbb{N}^N$ and a feasible tree solution y generated by T such that $b = \deg_T(N)$ and $c = \deg_T(P)$?

Towards a proof of Theorem 2





Towards a proof of Theorem 2



$$k \in \arg\min\{y_i \cdot \pi_{ij} \mid i \in P\}$$

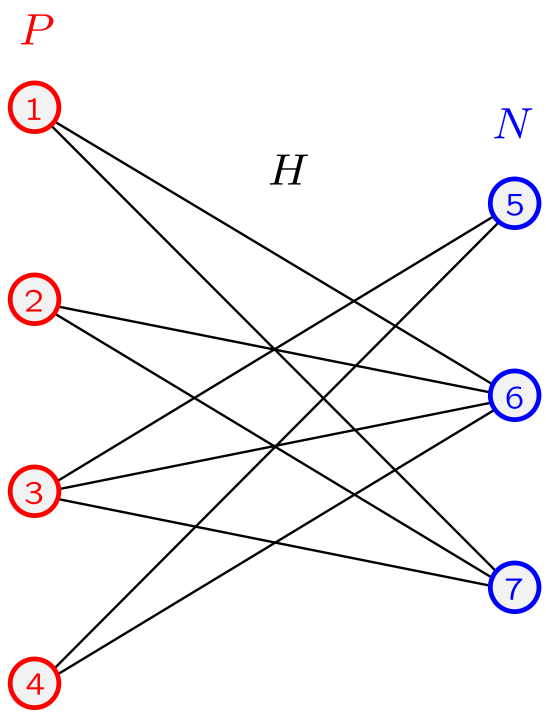
Towards a proof of Theorem 2

$\mathcal{T}(H)$ ... set of all spanning trees of a graph H

Lemma 1. Let $H = H(P, N)$ be a bi-partite graph. Then

$$|\{\mathrm{deg}_T(P) | \, T \in \mathcal{T}(H)\}| = |\{\mathrm{deg}_T(N) | \, T \in \mathcal{T}(H)\}|.$$

$$\begin{pmatrix} 1 & 2 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 1 \\ 3 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 & 2 & 2 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 4 & 3 & 3 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 & 3 & 2 \end{pmatrix}$$

Consequences of Theorem 1 and 2

Corollary 1. Assume that also (5) holds. Let x, y be two feasible tree solutions with respect to bi-partitions (P_x, N_x) and (P_y, N_y) of V, respectively. Then $(P_x, N_x) \neq (P_y, N_y)$ implies $x \neq \alpha y$ for all $\alpha > 0$. Moreover, K_d^+ has **at least** $2^d - 2$ extreme directions.

Corollary 2. K_d^+ has **at most** $\sum_{p=1}^{d-1} \binom{d-2}{p-1}\binom{d}{p}$ extreme directions.

Example. The upper bound in Corollary 2 cannot be improved.

Let the non-diagonal entries be pairwise different prime numbers such that

$$\left(\min\left\{\pi_{ij} |\ ij \in V \times V, i \neq j\right\}\right)^2 > \max\left\{\pi_{ij} |\ ij \in V \times V, i \neq j\right\}$$

Example. $d = 20$, $\pi_{ii} = 1$, $\pi_{12} = 59$, $\pi_{12} = 61$ ... $\pi_{20,19} = 2713$

$$59^2 > 2713 \implies (5)$$

K_{20}^{+} has exactly $\sum_{p=1}^{19} \binom{18}{p-1}\binom{20}{p} = 35.345.263.800$ extreme directions.

$P = \{5, 6, 7, 8, 9, 10, 11\}$, $N = \{1, \ldots, 4, 12, \ldots, 20\}$.

$\binom{d-2}{p-1} = \binom{18}{6} = 18564$ P-configurations for this bi-partition ($p := |P|$).

$c = (3, 2, 4, 2, 2, 2, 4)^T \in \mathbb{N}^P$

Algorithm (Matlab, about 15 minutes):

$$y = \Big(\tfrac{487\cdot757}{503\cdot859}, \tfrac{491\cdot757}{503\cdot859}, \tfrac{619\cdot947\cdot1367}{677\cdot953\cdot1427}, \tfrac{757}{859}, \tfrac{757}{503\cdot859}, \tfrac{947\cdot1367}{677\cdot953\cdot1427}, \tfrac{1}{859}, \tfrac{1367}{953\cdot1427},$$
$$\tfrac{1}{1117}, \tfrac{839}{859\cdot1237}, \tfrac{1}{1427}, \tfrac{1327}{1427}, \tfrac{947\cdot1367}{953\cdot1427}, \tfrac{1367}{1427}, \tfrac{1373}{1427}, \tfrac{829}{859}, \tfrac{839}{859}, \tfrac{839\cdot1249}{859\cdot1237}, \tfrac{1109}{1117}, 1\Big)^T$$

$b = (1, 1, 1, 2, 1, 2, 2, 1, 1, 2, 1, 1, 3)^T \in \mathbb{N}^N$

Special cases

$\pi_{ii} := 1$ and $\pi_{ij} := a_j/b_i$ $(i \neq j)$, $0 < b_i \leq a_i$ $(i \in V)$,
$0 < b_k < a_k$ for at least one $k \in V$

$$\Rightarrow \text{(1) to (4)}$$

[if $0 < b_i < a_i$ $(i \in V)$ $\Rightarrow$ (5)]

Then, every bi-partition yields only one feasible tree solution (and thus just one generating vector of K_d^+):

Corollary 3.

$K_d^+ = \text{cone}\left\{y \in \mathbb{R}^d | \ (P, N) \text{ bi-part. of } V, \ \forall i \in P : \ y_i = b_i, \ \forall j \in N : \ y_j = a_j\right\}$

K_d^+ has **at most** $2^d - 2$ extreme directions.
If (5) is satisfied, K_d^+ has **exactly** $2^d - 2$ extreme directions.

Special cases

Case 1: d currencies with positive bid-ask-spread.

d currencies with bid prices $b = (1, S_1, ..., S_d)$ and ask prices $a_i = (1+k)b_i$ for all i. Proportional transaction costs $k > 0$.
$\Rightarrow$ (5). $\Rightarrow$ **exactly** $2^d - 2$ extreme directions.

Recursive representation case 1:

For $d \geq 3$ (columns of Y_d are generating vectors of K_d^+):

$$Y_2 = \begin{pmatrix} a_1 & b_1 \\ b_2 & a_2 \end{pmatrix} \qquad Y_d = \begin{pmatrix} & & & b_1 & & & & a_1 \\ & Y_{d-1} & & \vdots & & Y_{d-1} & & \vdots \\ & & & b_{d-1} & & & & a_{d-1} \\ a_d & \dots & a_d & a_d & b_d & \dots & b_d & b_d \end{pmatrix}.$$

Note: $2^d - 2 = 2(2^{d-1} - 2) + 2$

Special cases

Case 2: d assets all denoted in domestic currency (= asset 1), exchanges only via domestic currency.

Recursive representation case 2:

bid and ask prices $b_i < a_i$ for $i \in \{2, ..., d\}$ expressed by asset 1 ('cash'). Since $a_1 = b_1 = 1$ (cash) $\Rightarrow$ (5) is not satisfied, $\Rightarrow$ less than $2^d - 2$ extreme directions.

For $d \geq 3$ (columns of Y_d are generating vectors of K_d^+)

$$Y_2 = \begin{pmatrix} 1 & 1 \\ a_2 & b_2 \end{pmatrix} \qquad Y_d = \begin{pmatrix} & Y_{d-1} & & & Y_{d-1} & \\ a_d & \dots & a_d & b_d & \dots & b_d \end{pmatrix}.$$

K_d^+ has exactly 2^{d-1} extreme directions.

Recall:

$\lvert K_d \rvert,\ \lvert K_d^+ \rvert$	$d=2$	3	4	...	7	...	d
general	2, 2	6, 6	12, 20	...	42, 924	...	$d(d-1)$, $\sum_{p=1}^{d-1} \binom{d-2}{p-1} \binom{d}{p}$
case 1	2, 2	6, 6	12, 14	...	42, 126	...	$d(d-1)$, $2^d - 2$
case 2	2, 2	4, 4	6, 8	...	12, 64	...	$2(d-1)$, 2^{d-1}

case 1: d currencies with positive bid-ask-spread.
case 2: d assets all denoted in domestic currency (= asset 1), exchanges only via domestic currency.

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Thank you!

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