Continuity Properties of Law-Invariant (Quasi-)Convex Risk Functions on L^∞

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Abstract

We study continuity properties of law-invariant (quasi-)convex functions $f : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to (-\infty, \infty]$ over a non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This is a supplementary note to [12].

Key words: law-invariant (quasi-)convex risk measures, duality, Fatou property.

1 Introduction and Statement of the Result

Throughout this note we assume that the reader is familiar with basic duality theory and related terminology as outlined in e.g. [2], [7] or [15]. Moreover, since this is meant to be a short supplementary text, we also presume knowledge of the concept of convex risk measures and related continuity problems as outlined in [10], [9], and in particular [12], the latter being the main reference of this note. In their seminal paper [12] Jouini, Schachermayer and Touzi prove that given any standard non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ every law-invariant convex risk measure $\rho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) =: L^\infty \to \mathbb{R}$ satisfies the Fatou property. Law-invariant means that $\rho(X) = \rho(Y)$ for any $X, Y \in L^\infty$ being identically distributed ($X \sim Y$). The Fatou property equals lower semi-continuity (l.s.c.) of ρ with respect to the $\sigma(L^\infty, L^1)$-topology and is desirable since it yields a dual representation of ρ as a supremum over weighted probabilities, i.e.

$$\rho(X) = \sup_{\mathbb{Q} \ll \mathbb{P}} E_{\mathbb{Q}}[-X] - \rho^*(\mathbb{Q}), \quad X \in L^\infty, \tag{1.1}$$

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where

$$\rho^*(\mathbb{Q}) := \sup_{X \in L^\infty} E_{\mathbb{Q}}[-X] - \rho(X)$$

for any probability measure $\mathbb{Q} \ll \mathbb{P}$ on $(\Omega, \mathcal{F})$. More generally, the results of [12] imply that given any standard non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$, any l.s.c. (with respect to the $\|\cdot\|_\infty$-topology) law-invariant convex function $f : L^\infty \to (-\infty, \infty]$ (not necessarily a convex risk measure) is automatically l.s.c. with respect to the $\sigma(L^\infty, L^1)$-topology. In this note we show that the assumption of standardness of the probability space can be dropped (see appendix A for a short survey on standardness, and example 1.4 for a non-standard probability space). This may be no surprise, it seems obvious that the standardness of the underlying probability space should not play a prominent role, however we feel the necessity of clarifying this fact since the main result of [12] is often cited (see [1], [3], [8], [9], [11], [13], [14], [18], and many more). Thus many of the results building on [12] in fact hold on general non-atomic probability spaces, which meets the common urge not to impose other requirements on the underlying probability space than that it be rich enough to support continuously distributed random variables, i.e. that it be non-atomic. Moreover, we also generalize to the quasi-convex case which is becoming increasingly popular (see e.g. [3] and [6], and [4] for a duality theory for quasi-convex functions). Our aim is to prove

Proposition 1.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a non-atomic probability space and let $f : L^\infty \to (-\infty, \infty]$ be a law-invariant l.s.c. (with respect to $\|\cdot\|_\infty$-topology) quasi-convex function, then f is l.s.c. with respect to any $\sigma(L^\infty, L^q)$-topology for every $q \in [1, \infty]$.*

So in case of a law-invariant convex risk measure ρ we may even obtain a dual representation (1.1) over probabilities with bounded densities $d\mathbb{Q}/d\mathbb{P}$. A version of proposition 1.1 for the standard case and a convex function has already been shown in [9]. Proposition 1.1 immediately follows from the following proposition 1.2 because l.s.c. of the function $f : L^\infty \to (-\infty, \infty]$ with respect to the $\sigma(L^\infty, L^q)$-topology is equivalent to the levels sets $E_k := \{X \in L^\infty \mid f(X) \leq k\}$, $k \in \mathbb{R}$, being closed in the $\sigma(L^\infty, L^q)$-topology, whereas quasi-convexity of f (i.e. $f(\lambda X + (1-\lambda)Y) \leq \max\{f(X), f(Y)\}$ for all $X, Y \in L^\infty$ and $\lambda \in [0, 1]$) is equivalent to convexity of the level sets.

Proposition 1.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a non-atomic probability space and let $C \subset L^\infty$ be a law-invariant (i.e. $X \in C$ and $Y \sim X$ implies $Y \in C$) convex set which is closed in the $\|\cdot\|_\infty$-topology ($\|\cdot\|_\infty$-closed). Then, C is closed with respect to any $\sigma(L^\infty, L^q)$-topology for every $q \in [1, \infty]$.*

The core of the proof of proposition 1.2 is the following lemma 1.3 which is the analog in our setting of lemma 4.2 of [12]. The idea of its proof is similar to the original proof for standard probability spaces as presented in [12], the difference being that we do not deal with measure preserving maps which do not necessarily exist if the underlying probability space is not assumed to be standard, but argue by means of the quantile function.

Lemma 1.3. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a non-atomic probability space and let $C\subset L^\infty$ be a law-invariant convex $\|\cdot\|_\infty$-closed set, then for all sub-σ-algebras $\mathcal{A}\subset\mathcal{F}$ and all $X\in C$ we have that $E[X\mid\mathcal{A}]\in C$.*

Proof. The proof follows the lines of the corresponding proof in [12].

Step 1: First of all we prove the assertion for the expectation, i.e. $\mathcal{A}=\{\emptyset,\Omega\}$. Let $X\in C$ and $\epsilon>0$. Without loss of generality we may assume that $X\geq 0$ (otherwise we shift C by a constant). Denote by

$$q_X:[0,1]\to\mathbb{R},\quad q_X(s):=\begin{cases}\operatorname{essinf}X & s=0,\\ \inf\{x\mid\mathbb{P}(X\leq x)\geq s\} & s\in(0,1),\\ \operatorname{esssup}X & s=1\end{cases}$$

the (left-continuous) quantile function of X. Note that $0\leq q_X(s)\leq q_X(1)=\|X\|_\infty<\infty$, so the quantile function is bounded. Let $n\in\mathbb{N}$ be such that $\frac{q_X(1)}{n}\leq\epsilon$, and let $A_i:=(\frac{i-1}{n},\frac{i}{n}]$ for $i=1,\dots,n-1$ and $A_n:=(1-\frac{1}{n},1)$. Moreover, let $B_1,\dots,B_n$ be a partition of Ω such that $\mathbb{P}(B_k)=1/n$ (this exists according to theorem 9.51 of [2]). Since $(\Omega,\mathcal{F},\mathbb{P}(\cdot\mid B_k))$ is a non-atomic probability space for every $k\in\{1,\dots,n\}$, for any $j\in\{1,\dots,n\}$ there exists a random variable U_j^k which is uniformly distributed on A_j under $\mathbb{P}(\cdot\mid B_k)$, i.e. $\mathbb{P}(U_j^k\leq s\mid B_k)=ns-j+1$, $s\in A_j$ (proposition A.27 of [10]). Then, for any permutation $\pi:\{1,\dots.n\}\to\{1,\dots.n\}$, the random variable $U^\pi:=\sum_{k=1}^n U^k_{\pi(k)}1_{B_k}$ is uniform on $(0,1)$ under $\mathbb{P}$. Hence $X^\pi:=q_X(U^\pi)=\sum_{k=1}^n q_X(U^k_{\pi(k)})1_{B_k}$ has the same distribution as X and is thus an element of C. As C is convex we infer that

$$X_n:=\frac{1}{n!}\sum_{\pi\in S_n}X^\pi\in C$$

where S_n denotes the set of all permutations $\pi:\{1,\dots.n\}\to\{1,\dots.n\}$. Moreover, we have (by monotonicity of the quantile function) that

$$\begin{aligned}\|X_n-E[X]\|_\infty &= \|X_n-\int_0^1 q_X(s)ds\|_\infty \leq \frac{1}{n}\sum_{i=1}^n\Big(q_X(\frac{i}{n})-q_X(\frac{i-1}{n})\Big)\\ &\leq \frac{q_X(1)}{n} \leq \epsilon.\end{aligned}$$

Since $\epsilon>0$ was arbitrary and C is closed, we conclude that $E[X]\in C$.

Step 2: Now we continue as in [12] and argue for the case $\mathcal{A}=\sigma(D_1,\dots,D_r)$ for some finite partition $D_1,\dots,D_r\in\mathcal{F}$ of Ω with $\mathbb{P}(D_i)>0$ for all $i=1,\dots,r$, $r\in\mathbb{N}$. Let $X\in C$. We consider the non-atomic probability spaces $(D_i,\mathcal{F}_i,\mathbb{P}_i)$ where $\mathcal{F}_i:=\{A\cap D_i\mid A\in\mathcal{F}\}$ and $\mathbb{P}_i:\mathcal{F}_i\to[0,1]$, $A\mapsto\mathbb{P}(A)/\mathbb{P}(D_i)$, and the law-invariant convex sets $C_i:=\{Y_{|D_i}\mid Y\in C\}\subset L^\infty(D_i,\mathcal{F}_i,\mathbb{P}_i)$, $i=1,\dots,r$. Given any $\epsilon>0$, by the same arguments as in step 1 applied to C_i over $(D_i,\mathcal{F}_i,\mathbb{P}_i)$, we obtain some $X_n^i=\frac{1}{n!}\sum_{\pi\in S_n}X^{i,\pi}\in C_i$ such that $X^{i,\pi}$ and $X_{|D_i}$ are identically distributed under $\mathbb{P}_i$ for every $\pi\in S_n$ and $\|X_n^i-E_{\mathbb{P}_i}[X_{|D_i}]\|_\infty\leq\epsilon$.

(We may assume that the n is independent of i because when applying step 1 to C_i we may always choose the maximal "n over all $i = 1, \dots, r$") Let

$$X_n := \sum_{i=1}^{r} X_n^i 1_{D_i} = \frac{1}{n!} \sum_{\pi \in S_n} \sum_{i=1}^{r} X^{i,\pi} 1_{D_i}.$$

Then $X_n \in C$, since $\sum_{i=1}^{r} X^{i,\pi} 1_{D_i}$ and X are identically distributed under $\mathbb{P}$. Moreover, we have that

$$\|X_n - E[X|\mathcal{A}]\|_\infty = \|X_n - \sum_{i=1}^{r} E_{\mathbb{P}_i}[X_{|D_i}] 1_{D_i}\|_\infty \le \epsilon.$$

As $\epsilon > 0$ was arbitrary and as C is closed, we conclude that $E[X|\mathcal{A}] \in C$.

Step 3: For the final step we note that for any sub-σ-algebra $\mathcal{A} \subset \mathcal{F}$ there is a sequence of finite sub-σ-algebras $\mathcal{A}_n \subset \mathcal{F}$, $n \in \mathbb{N}$, such that $\|E[X|\mathcal{A}_n] - E[X|\mathcal{A}]\|_\infty \to 0$ for $n \to \infty$, so the assertion follows from step 2 and closedness of C. □

Now the proof of proposition 1.2 is literally the same as the proof of lemma 2.4 (i) in [9]. For the sake of completeness we repeat it here.

Proof of proposition 1.2. Step 1: Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. As in [12] we define the conditional expectation on $L^{\infty *}$ (the dual space of $(L^\infty, \|\cdot\|_\infty)$) as a function $E[\cdot \mid \mathcal{G}] : L^{\infty *} \to L^{\infty *}$ where $E[\mu \mid \mathcal{G}]$ is given by

$$\langle E[\mu \mid \mathcal{G}], X \rangle := \langle \mu, E[X \mid \mathcal{G}] \rangle \quad \forall\, X \in L^\infty .$$

If $\mathcal{G}$ is finite, i.e. $\mathcal{G} = \sigma(A_1, \dots, A_n)$ for a finite partition $A_1, \dots, A_n \in \mathcal{F}$ of Ω with $\mathbb{P}(A_i) > 0$ for all $i = 1, \dots, n$, then $E[\mu \mid \mathcal{G}] \in L^\infty$ since for all $X \in L^\infty$ we have

$$\langle E[\mu \mid \mathcal{G}], X \rangle = \langle \mu, E[X \mid \mathcal{G}] \rangle = \sum_{i=1}^{n} E[X 1_{A_i}] \frac{\mu(A_i)}{\mathbb{P}(A_i)}.$$

Hence, $E[\mu \mid \mathcal{G}] = \sum_{i=1}^{n} \frac{\mu(A_i)}{\mathbb{P}(A_i)} 1_{A_i} \in L^\infty$.

Step 2: If $C = \emptyset$, the assertion of proposition 1.2 is obvious. For the remainder of this proof, we assume that $C \neq \emptyset$. Now let $(X_i)_{i \in I}$ be a net in C converging to some $X \in L^\infty$ in the $\sigma(L^\infty, L^q)$-topological sense, i.e. $E[ZX_i] \to E[ZX]$ for all $Z \in L^q$. Then, in view of step 1, if $\mathcal{G}$ is finite, we have $E[E[\mu \mid \mathcal{G}]X_i] \to E[E[\mu \mid \mathcal{G}]X]$ for all $\mu \in L^{\infty *}$. But this equals $\langle \mu, E[X_i \mid \mathcal{G}] \rangle \to \langle \mu, E[X \mid \mathcal{G}] \rangle$ for all $\mu \in L^{\infty *}$. In other words, the net $(E[X_i \mid \mathcal{G}])_{i \in I}$ converges to $E[X \mid \mathcal{G}]$ in the $\sigma(L^\infty, L^{\infty *})$-topology. Since, according to lemma 1.3, $E[X_i \mid \mathcal{G}] \in C$ for all $i \in I$, we conclude that $E[X \mid \mathcal{G}] \in C$, because C is closed and convex and thus $\sigma(L^\infty, L^{\infty *})$-closed. Hence, $E[X \mid \mathcal{G}] \in C$ for all finite sub-σ-algebras $\mathcal{G} \subset \mathcal{F}$. Recalling that we can approximate X in $(L^\infty, \|\cdot\|_\infty)$ by a sequence of conditional expectations $(E[X \mid \mathcal{G}_n])_{n \in \mathbb{N}}$ in which the $\mathcal{G}_n$ are all finite, we conclude by means of the $\|\cdot\|_\infty$-closedness of C that $X \in C$. Thus C is $\sigma(L^\infty, L^q)$-closed. □

Example 1.4. We give a simple example of a non-atomic probability space which is not standard. To this end, consider the non-atomic standard probability space per se which is $([0,1], \mathcal{B}([0,1]), \lambda)$ where $\mathcal{B}([0,1])$ denotes the Borel-σ-algebra on $[0,1]$ and λ is the Lebesgue measure. Now let $\Omega := [0,1] \times \{0,1\}$ and denote by π the projection of Ω onto $[0,1]$. We construct a probability space by equipping Ω with the σ-algebra $\mathcal{F} := \{\pi^{-1}(B) | B \in \mathcal{B}([0,1])\}$, and with the probability measure $\mathbb{P}(A) := \lambda(\pi(A))$, $A \in \mathcal{F}$. Then, $(\Omega, \mathcal{F}, \mathbb{P})$ is non-atomic, but not standard since there exists no measurable bijection $f : \Omega \to [0,1]$.

A Standard Probability Space

Two probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{B}, \mathbb{Q})$ are *isomorphic mod 0* if there exists null-sets $A \in \mathcal{F}$ and $B \in \mathcal{B}$ and a bijection $f : \Omega \backslash A \to \Omega' \backslash B$ such that both f and f^{-1} are measurable and measure-preserving (i.e. $\mathbb{P}(C \cap A^c) = \mathbb{Q}(f(C \cap A^c))$ for all $C \in \mathcal{F}$) on the restricted probability spaces. The map f is called *isomorphism mod 0*. An non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is *standard* if it is isomorphic mod 0 to the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$ where $\mathcal{B}([0,1])$ denotes the Borel-σ-algebra over $[0,1]$ and λ is the Lebesgue-measure on $\mathcal{B}([0,1])$ (see [16] section 2). A mapping $\tau : \Omega \to \Omega$ is a *measure preserving transformation* if it is an isomorphism mod 0. Given an non-atomic standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two sets $A, B \in \mathcal{F}$ such that $\mathbb{P}(A) = \mathbb{P}(B)$, there exists a measure preserving transformation $\tau : \Omega \to \Omega$ such that $\tau(A) = B$ $\mathbb{P}$-a.s. and $\tau(B) = A$ $\mathbb{P}$-a.s. and $\tau = \mathrm{Id}_\Omega$ on $A^c \cap B^c$ $\mathbb{P}$-a.s. This is a direct consequence of the definition of standardness and the fact that for every subset $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$ the restricted probability space with conditional measure is again standard (see [16] section 2, in particular 2.3 and 2.4). For instance, if Ω is a complete separable metric space, $\mathcal{F}$ the corresponding σ-algebra of Borel-sets, and $\mathbb{P}$ a probability measure on $(\Omega, \mathcal{F})$, then $(\Omega, \mathcal{F}, \mathbb{P})$ is standard (see e.g. [17] theorem 9, p. 327).

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