Nonrecursive Separation of Risk and Time Preferences

Matthias Albrecht Fahrenwaldt[*], Ninna Reitzel Jensen, Mogens Steffensen[†]

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Abstract

Recursive utility disentangles preferences with respect to time and risk by recursively building up a value function of local increments. This involves certainty equivalents of indirect utility. Instead we disentangle preferences with respect to time and risk by building up a value function as a non-linear aggregation of certainty equivalents of direct utility of consumption. This entails time-consistency issues which are dealt with by looking for an equilibrium control and an equilibrium value function rather than a classical optimal control and a classical optimal value function. We characterize the solution in a general diffusive incomplete market model and find that, in certain special cases of utmost interest, the characterization coincides with what would arise from a recursive utility approach. But also importantly, in other cases, it does not: The two approaches are fundamentally different but match, exclusively but importantly, in the mathematically special case of homogeneity of the value function.

Keywords: Time-consistency, time-global preferences, recursive utility, equilibrium strategies, generalized Hamilton–Jacobi–Bellman equation, continuous time, certainty equivalents.

1 Introduction

We formulate a continuous-time dynamic consumption-investment problem where preferences with respect to risk and time variability are disentangled. In contrast to recursive utility which also builds on the idea of disentangling these preferences, our value function is based on a time-global objective with non-time-additive utility. This allows for working with certainty equivalents of direct utility of consumption rather than indirect utility. Time-inconsistency arising from non-time-additivity is dealt with by looking for a subgame perfect equilibrium among a continuum of selves. We consider a general incomplete market with coefficients driven by a non-hedgeable economic state process. In special cases that include the Merton market, we find a resulting behavior that coincides with that coming from recursive utility with Epstein-Zin preferences. Among these special cases, we also find closed-form solutions to new problem formulations beyond standard power utility, including non-hedgeable consumer price indexation and exponential utility. Thus, our contribution to the literature is two-fold:

We base—we believe as the first—the disentanglement on a time-global objective for a general financial market. Second we detect new and relevant solveable consumption-investment problems in incomplete markets within our problem formulation where the solution coincides with what would have been obtained by recursive utility.

Recursive utility was developed by Epstein and Zin (1989, 1991) based on work by Kreps and Porteus (1978, 1979). It is celebrated for disentangling preferences with respect to risk and time. Its continuous-time limit, spoken of as stochastic differential utility or, simply, continuous-time recursive utility, developed by Duffie and Epstein (1992b) has the same ability to allow for separate preference functions against variability over risk and (continuous) time. It is widely used to study optimal consumption-portfolio choice in various markets, see e.g. Schroder and Skiadas (1999, 2005); Kraft et al. (2013). Also, recursive utility is used to examine ambiguity aversion and preferences for resolution of uncertainty, see e.g. Chen and Epstein (2002); Skiadas (1998, 2013). Issues with differentiability when going to continuous-time were addressed by Kraft and Seifried (2010, 2014). A particularity of recursive utility is, of course, the definitional recursive building of the value function or indirect utility function. This means that, when locally aggregating present consumption with the utility of future consumption, the latter is represented by its indirect utility. In the recursion appears the certainty equivalent with respect to the representative indirect utility of wealth rather than the underlying future uncertain consumption.

Indirect utility appears to be the right representative of utility of future consumption, given that we start out with a recursive definition. Yet, here we suggest to start out with a time-global objective built up by certainty equivalents with respect to future uncertain consumption. Said differently, we suggest to replace the indirect utility representation of future consumption by the direct utility of future consumption itself. Apart from that, our objective remains the same: To separate preferences for risk and time. Once having formed certainty equivalents of future consumption at different points in time, we think of them as "certain" values attributed to these time points. This allows for a non-linear aggregation of these certainty equivalents which relates to preferences with respect to time only. Our objective becomes non-linear in time which, at first sight, dumps the idea for reasons of time-inconsistency issues that are completely avoided with recursive utility. There, the controls are, definitional from the recursive structure, time-consistent, so why bother with time-inconsistency issues? Because, we find the construction of a time-global objective based on direct utility of future consumption instead of indirect utility appealing and, by now, the complications with time-inconsistency can be overcome. That is, because we should and because we can.

Already in the definition of recursive utility, time-consistency issues are delicately avoided. First the certainty equivalent of the indirect utility is formed. Then this is non-linearly time-aggregated with present consumption. The alternative order is unfriendly: To first non-linearly time-aggregate indirect utility and consumption and then take the expected utility here-of. It is the non-linear time-aggregation under uncertainty that leads to time-inconsistency issues. Although we suggest a completely different formulation, we also have time-inconsistency issues, but for different reasons. We make non-linear time-aggregation of objects we can think of as certain like it is done for recursive utility. But we aggregate over a global time-horizon rather than a local (one-period in discrete-time and infinitesimal in continuous-time) time-horizon, as in the case of recursive utility.

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Time-inconsistent behavior was initially formalized by Strotz (1955). Pollak (1968), Goldman (1980), and Laibson (1997) contributed to the understanding of the problem as an intra-personal game and looked for subgame perfect equilibria. Ekeland and Pirvu (2008) defined a continuous-time subgame perfect equilibrium in order to deal with the time-inconsistency arising from replacing exponential discounting of utility by hyperbolic discounting. We follow their definition and derive the equilibrium value and equilibrium strategy when the time-inconsistency arises from the non-linear aggregation of certainty equivalents as explained above.

The idea of summing up certainty equivalents over global time was also pursued by Jensen and Steffensen (2015). They considered a consumption-investment-insurance problem in a Merton market for an investor with an uncertain lifetime and access to life insurance. The disentanglement of preferences for risk and time is, there, a starting point for the idea of also disentangling utility of consumption as alive and inheritors utility of consumption after the death of the investor. Already they show that in the special case of a Merton market the solution to our optimization problem coincides with that of recursive utility with Epstein-Zin preferences. We obtain the coincidence with the Merton market and recursive utility from a different angle.

We start out with a general diffusive, incomplete market with a risky, diffusive asset with price coefficients driven by another diffusive economic state process that cannot be perfectly hedged. Coincidence with the solution for recursive utility with Epstein-Zin preferences in the special case of a Merton market falls out. But we also characterize solutions for much more general markets that have previously been studied under recursive utility. This unveils, in terms of resulting behavior, a fundamental difference between recursive utility and our approach. In general cases, studied under recursive utility by Chacko and Viceira (2005) and Kraft et al. (2013), the generalized Bellman equation that we find to characterize our equilibrium value, contains additional terms compared to the standard recursive utility Bellman-type equation. Only when we have complete separability in time, wealth, and the economic state process, we agree with users of recursive utility on the characterization of the solution. On the other hand, we study in details such special cases leading to linearly homogeneous value functions that, to our knowledge, have not been studied before. They include cases with power utility where we scale consumption by the economic state process, interpreting this process as an only partly hedgeable consumer price index, and cases with exponential utility. We provide explicit solutions in these cases.

The outline of the paper is as follows. In Section 2, we present the model for the price and wealth processes. We motivate our problem formulation and relate it to standard recursive utility. In Section 3, we define the set of admissible controls and the concept of equilibrium and state our main theorem with sufficient conditions to determine equilibrium controls and the corresponding equilibrium value function. In Section 4, we present two non-trivial examples of the framework with incomplete markets. We consider two different choices of the utility functions, namely power utility and exponential utility. We provide explicit solutions, and we establish a connection to recursive utility.

2   General Set-Up and Optimization Problem

In this section, we present our optimization problem and its connection to related problems. This section is central because it is our problem formulation, rather than

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the solution, that is the innovative part of the paper.

We consider an investor making decisions concerning consumption, c, and investment, π, in a Brownian market. The wealth of the investor evolves according to the dynamics

$$(1) \qquad \mathrm{d}X_t^{c,\pi} = \mu^{c,\pi}\left(t, X_t^{c,\pi}, Y_t\right)\mathrm{d}t + \sigma^{c,\pi}\left(t, X_t^{c,\pi}, Y_t\right)\mathrm{d}W_t \ , \ X_0^{c,\pi} = x_0 \ ,$$

where Y is a non-traded state process with the dynamics

$$(2) \qquad \mathrm{d}Y_t = \alpha\left(t, Y_t\right)\mathrm{d}t + \beta\left(t, Y_t\right)\left(\rho\,\mathrm{d}W_t + \sqrt{1-\rho^2}\,\mathrm{d}\bar{W}_t\right) \ , \ Y_0 = y_0 \ .$$

Here, $\mu^{c,\pi}, \sigma^{c,\pi}, \alpha, \beta$ are sufficiently regular functions, and W and $\bar{W}$ are two independent Brownian motions. The volatility in $X^{c,\pi}$ arise from investment in a stock. For later use, we introduce the infinitesimal generator $\mathcal{A}^{c,\pi}$ of $(X^{c,\pi}, Y)$ which is given by

$$\mathcal{A}^{c,\pi} = \mu^{c,\pi}\partial_x + \frac{1}{2}\left(\sigma^{c,\pi}\right)^2\partial_x^2 + \alpha\partial_y + \frac{1}{2}\beta^2\partial_y^2 + \rho\beta\sigma^{c,\pi}\partial_{xy} \ .$$

Note that this operator is both time- and space-dependent. In the next section, we make assumptions concerning the coefficients to guarantee the existence and uniqueness of solutions to the Kolmogorov equations $\partial_t l = -\mathcal{A}^{c,\pi}l$ with appropriate terminal condition. Also, our main theorem will assume the existence of such a solution. This has to be checked in concrete situations.

A classical optimization problem formalized for the investor is that of maximizing expected time-additive utility of consumption and final wealth,

$$(3) \qquad \sup_{c,\pi}\mathbb{E}\left[\int_0^T e^{-\delta s}u\left(c\left(t, X_t^{c,\pi}, Y_t\right)\right)\mathrm{d}t + e^{-\delta T}\Phi\left(X_T^{c,\pi}\right)\right] \ ,$$

where $\delta \geq 0$ is a subjective utility discount rate, u is an instantaneous utility function, and Φ is a utility function for final wealth. The utility functions u and Φ characterize the investor's preferences with respect to risk. The problem in (3) can be dealt with by embedding it in a value function given by

$$(4) \qquad V(t, x, y) = \sup_{c,\pi}\mathbb{E}_{t,x,y}\left[\begin{array}{c}\int_t^T e^{-\delta(s-t)}u\left(c\left(s, X_s^{c,\pi}, Y_s\right)\right)\mathrm{d}s \\ +e^{-\delta(T-t)}\Phi\left(X_T^{c,\pi}\right)\end{array}\right] \ ,$$

where $\mathbb{E}_{t,x,y}$ denotes conditional expectation given $X_t^{c,\pi} = x$ and $Y_t = y$. The controls (c, π) are chosen among a set of admissible strategies which essentially means that (1) has a solution and that certain integrals with respect to the Brownian motions have expectation zero. By means of dynamic programming techniques, the value function can be characterized by a certain partial differential equation containing a local optimization problem at each point (t, x, y). The solution for (c, π) to the local optimization problem can be proven to also produce the solution for (c, π) to the global optimization problem in (3). This is essentially a consequence of the linearity of the expectation operating on an infinitesimal sum of utility of consumption rates. We spell out here that this linearity is essential for the coincidence between local and global optimization since this linearity is serially spoiled below—and, thus, is also the coincidence.

We wrote above that the utility function u characterizes the investor's preferences with respect to risk, but u also plays a different indirect role in the time-additivity of

(4). Below, we formalize a way to disentangle preferences for risk and time, but, first, we consider briefly how the disentanglement is typically established within the theory of recursive utility. Instead of working with time-global objectives like the one given in (4), the standard approach in recursive utility is to study the local discrete-time objective

(5) $$V(t, X_t) = W\left(c_t, u^{-1}\left(\mathbb{E}_t\left[V\left(t + \Delta, X_{t+\Delta}\right)\right]\right)\right),$$

where c_t is the consumption at time t, $u^{-1}\left(\mathbb{E}_t\left[V\left(t + \Delta, X_{t+\Delta}\right)\right]\right)$ is the time t certainty equivalent of having $X_{t+\Delta}$ for consumption from time $t + \Delta$ and onward, and the function W is the so-called time aggregator, aggregating the utility of present consumption c_t and future consumption represented by $u^{-1}\left(\mathbb{E}_t\left[V\left(t + \Delta, X_{t+\Delta}\right)\right]\right)$. An important special case is when $(c, v) \mapsto W\left(c, u^{-1}(v)\right)$ is additive in c and v. Then we obtain time-additive utility, see e.g. Duffie and Epstein (1992b).

The continuous-time equivalent of these patterns of thinking were studied by Duffie and Epstein (1992a,b) under the name stochastic differential utility. The main ingredients are still a certainty equivalent of the value function (indirect utility) and an aggregator. However, taking $\Delta \to 0$ in (5) is, in general, a complicated operation that involves differentiability of the certainty equivalent and the aggregator. Kraft and Seifried (2010) propose an alternative notion of differentiability compared to Duffie and Epstein (1992a,b) in order to make the notion of stochastic differential utility more general and robust to e.g. inclusion of non-Brownian markets.

In (5), we form a so-called certainty equivalent in terms of

$$u^{-1}\left(\mathbb{E}_t\left[V\left(t + \Delta, X_{t+\Delta}\right)\right]\right),$$

i.e. in terms of the indirect utility V of wealth rather than the utility function u of consumption. Below we formalize a problem that is based on the certainty equivalence of direct utility of consumption rather than indirect utility. The fundamental idea is to formalize a continuous-time global optimization problem that encompasses both risk and time preferences. By working directly in continuous-time, we immediately obtain the benefits of continuous-time tractability without facing the differentiability issues arising when Δ tends to zero in (5). Our approach is in sharp contrast to the approach by Duffie and Epstein (1992a,b) and Kraft and Seifried (2010) who are certainly challenged by the notion of differentiability. We suggest the following approach:

For each future time point s, we form the certainty equivalent of the consumption rate, conditional on $X^{c,\pi}(t) = x$ and $Y(t) = y$,

$$u^{-1}\left(\mathbb{E}_{t,x,y}\left[u\left(c\left(s, X_s^{c,\pi}, Y_s\right)\right)\right]\right).$$

For all $s > t$, these are known at time t, and we are therefore inclined to treat them as deterministic future consumption rates. Now, we let a different function, say $\bar{\varphi}$, formalize the investor's time preferences with respect to these certainty equivalents.

The investor's utility from time t and onward is

$$\int_t^T e^{-\delta(s-t)} \bar{\varphi} \left(u^{-1} \left(\mathbb{E}_{t,x,y} \left[u \left(c \left(s, X_s^{c,\pi}, Y_s \right) \right) \right] \right) \right) \, \mathrm{d}s$$

$$+ \omega e^{-\delta(T-t)} \bar{\varphi} \left(u^{-1} \left(\mathbb{E}_{t,x,y} \left[u \left(X_T^{c,\pi} \right) \right] \right) \right)$$

$$= \int_t^T e^{-\delta(s-t)} \varphi \left(\mathbb{E}_{t,x,y} \left[u \left(c \left(s, X_s^{c,\pi}, Y_s \right) \right) \right] \right) \, \mathrm{d}s$$

$$+ \omega e^{-\delta(T-t)} \varphi \left(\mathbb{E}_{t,x,y} \left[u \left(X_T^{c,\pi} \right) \right] \right) \, ,$$

where $u\left(X_T^{c,\pi}\right)$ is utility from final wealth, ω is a scaling factor allowing for different weight on utility from consumption and final wealth, and $\varphi = \bar{\varphi} \circ u^{-1}$.

At this point, it is clear that we have a problem beyond what can be dealt with by classical dynamic programming. Namely, due to the transform φ of the expectation, we cannot exploit the linearity of the expectation operator and interchange expectation and time-addition. Before discussing what we can do "instead of" classical dynamic programming, we twist the problem in three ways. First, we allow the utility of consumption and terminal wealth to depend on the process Y. More specifically, we replace $u\left(c\left(s, X_s^{c,\pi}\right)\right)$ by $u\left(Y_s, c\left(s, X_s^{c,\pi}\right)\right)$. This turns out to be mathematically tractable, and we can, for example, think of Y as an index of purchasing power or a minimum subsistence level, depending on the shape of u. Second, we introduce separate utility functions for consumption, u_1, and final wealth, u_2. Third, while we are at "destroying" the workability of dynamic programming techniques, we multiply the problem with the constant δ and transform it with an increasing function f. Now, the value function reads

(6) $$V^{c,\pi}(t,x,y) = f \left(\begin{array}{c} \int_t^T \delta e^{-\delta(s-t)} \left[\varphi \circ m_1^{c,\pi} \right] (t,s,x,y) \, \mathrm{d}s \\ + \omega \delta e^{-\delta(T-t)} \left[\varphi \circ m_2^{c,\pi} \right] (t,T,x,y) \end{array} \right) ,$$

where, for $0 \leq t \leq s \leq T$,

$$m_1^{c,\pi}(t,s,x,y) = \mathbb{E}_{t,x,y} \left[u_1 \left(Y_s, c \left(s, X_s^{c,\pi}, Y_s \right) \right) \right] \, ,$$
$$m_2^{c,\pi}(t,T,x,y) = \mathbb{E}_{t,x,y} \left[u_2 \left(Y_T, X_T^{c,\pi} \right) \right] \, .$$

The function f is convenient since it does not hurt optimal behavior such that we can choose f however we want to make the mathematical representation of the value function as attractive as possible. At the moment, the reader may think that $f = \mathrm{id}$ (the identity function) is a particularly attractive choice of f, but this turns out not to be true in general. Rather, one should seek a function f that, in some sense that is made clear in the following section, offsets the complication from the function φ under the integral. In an abstract sense, we seek a non-linearizing function f that offsets the non-linearity stemming from the function φ such that the problem of optimizing (6) appears linear.

The choice $f = \varphi^{-1}$ turns out to be particularly convenient, at least in some cases. This choice is motivated by calculations and remarks in the next section. Note that given this insight, the choice $f = \varphi^{-1} = \mathrm{id}$, corresponding to $\bar{\varphi} = u^{-1}$, shows why there is typically no "normalization issue" for time-additive utility. In that case the normalized value function is, indeed, given by (4).

The problem of maximizing (6) is certainly non-standard due to its serial non-linearity. A seemingly different and new strand of literature on non-linear optimization

problems was initiated recently by Basak and Chabakauri (2010). In a Brownian market, they solve the dynamic mean-variance problem without so-called precommitment. The combination of no precommitment and the variance appearing in the objective forms the non-linearity since the variance contains the non-linear square function of the expectation of the wealth. Recent works elaborate on the techniques: Björk et al. (2014) study the mean-variance investment problem in a general Markovian setting; Czichowsky (2013) works with mean-variance problems in a non-Markovian setting by means of quadratic projection methods; Kronborg and Steffensen (2015) study mainly the mean-variance problem in a Black-Scholes setting but include optimization over consumption. It turns out that these techniques are well-suited for approaching non-linear problems like (6) in a specific way. The idea of adding up certainty equivalents is already explored in Jensen and Steffensen (2015). There, focus is on disentanglement of risk aversion and EIS for a power maximizing investor with uncertain lifetime and access to life insurance in a Black-Scholes setting.

The problem of maximizing (6) is complicated and, since dynamic programming does not work, there is no reason to believe that solutions to local and global optimization problems coincide in the same way as for (3) and (4). The non-linearity of (6) means that the solution at time 0 is likely to be inconsistent with the solution at time $t > 0$ if we search for an optimal control among all the usually admissible ones, namely those for which (1) has a solution. By inconsistent we mean that the decision we make at time t is not the same as the decision we plan to make at time t, for the same realization of $(X^{c,\pi}, Y)$. Here, we proceed as in Jensen and Steffensen (2015), take inspiration from Björk et al. (2014), and search for an equilibrium control for the value function $V^{c,\pi}$. The theoretical background for the equilibrium approach is equilibrium theory of continuous-time games. Actually, the resulting strategy is a Nash equilibrium strategy in a game where infinitesimally many so-called multiple selves are competing and where the time t-self knows that the continuum of all "later selves", i.e. s-selves for $s > t$, face the same game. Therefore, Björk et al. (2014) speak of the resulting strategies as equilibrium strategies rather than optimal strategies. This approach produces an optimal control process that does not solve for the supremum over all usual strategies in a usual sense. Rather, it is the best strategy given that one will later on follow a strategy based on the same objective conditioning on updated information. This conforms with with Basak and Chabakauri (2010) and subsequent papers mentioned above. It should be mentioned that this approach to dynamic decision making dates further back to Strotz (1955), Pollak (1968) and Selten (1975).

3 Equilibrium and Verification Theorem

In this section we define the set of admissible controls and the concept of equilibrium and state our main theorem. The theorem gives sufficient conditions to determine the equilibrium controls and the equilibrium value function.

In the following, let f and φ be in C^2 and f^{-1} be in C^0. For technical reasons in the proof of Theorem 3.5, we must make a hypothesis that guarantees the existence and uniqueness of $C^{1,2}$-solutions to the terminal value problem

$$
(7) \qquad \left.\begin{aligned}
\partial_t u(t,x,y) &= -\mathcal{A}^{c,\pi} u(t,x,y) \;, \\
u(T,x,y) &= g(x,y) \;.
\end{aligned}\right\}
$$

Recall that we call a function $\mathbb{R}^n \to \mathbb{R}^m$ *slowly increasing* if it is smooth and all derivatives are bounded in norm by a polynomial.

Hypothesis 3.1. The matrix-valued function $[0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ given by

$$\begin{pmatrix} \sigma^{c,\pi}(t, x, y) & \rho\beta(t, y)\sigma^{c,\pi}(t, x, y) \\ \rho\beta(t, y)\sigma^{c,\pi}(t, x, y) & \beta(t, y) \end{pmatrix}$$

and the vector-valued function $[0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ given by $(\mu^{c,\pi}(t, x, y), \alpha(t, y))$ is smooth, have bounded first derivatives, and is slowly increasing. Also, $g \in C^3(\mathbb{R}^2)$ is polynomially bounded.

This hypothesis is not the most general one can make but is general enough for practical situations. We took these assumptions from Theorem 2.12 of Stroock (1983). The existence and uniqueness of classical solutions of (7) is then guaranteed by Theorem 2.21 of Stroock (1983). Given concrete choices of the coefficient functions of $\mathcal{A}^{c,\pi}$ one must check that (7) does indeed have a solution for any admissible controls.

Definition 3.2 (Admissible controls). *We call a control (c, π) admissible if*

(i) $(c, \pi) \in C([0, T] \times \mathbb{R}^2) \times C([0, T] \times \mathbb{R}^2)$,

(ii) Hypothesis 3.1 is satisfied.

Remark 3.3. Note that Hypothesis 3.1 ensures that the system of stochastic differential equations (SDE's) in Equations (1)–(2) has a unique solution by Theorem 2.12 of Stroock (1983). This solution has the property that $\mathbb{E}\left[\|(X_t, Y_t)\|^p\right] < \infty$ for all $p \in [2, \infty)$ for given initial conditions (x_0, y_0).

Rewriting Definition 2.1 in Björk et al. (2014) in the language of this paper, we get the following definition of equilibrium:

Definition 3.4. *Consider an admissible control (c^*, π^*) (informally viewed as a candidate equilibrium control). Choose a fixed, admissible control $(\bar{c}, \bar{\pi})$, a real number $h > 0$, and an initial point $(u, x, y) \in [0, T] \times \mathbb{R}^2$. Define the control $\left(c^h, \pi^h\right)$ by*

$$\left(c^h, \pi^h\right)(t, \bar{x}, \bar{y}) = \begin{cases} (\bar{c}, \bar{\pi})(t, \bar{x}, \bar{y}), & u \le t < u + h, \bar{x}, \bar{y} \in \mathbb{R}, \\ (c^*, \pi^*)(t, \bar{x}, \bar{y}), & u + h \le t \le n, \bar{x}, \bar{y} \in \mathbb{R}. \end{cases}$$

If for all admissible controls $(\bar{c}, \bar{\pi})$ and all points $(u, x, y) \in [0, T] \times \mathbb{R}^2$

$$(8) \qquad \liminf_{h \to 0} \frac{V^{c^*, \pi^*}(u, x, y) - V^{c^h, \pi^h}(u, x, y)}{h} \ge 0,$$

we say that (c^*, π^*) is an equilibrium control for the function $V^{c, \pi}$. The corresponding equilibrium value function V^* is given by

$$V^*(t, x, y) = V^{c^*, \pi^*}(t, x, y).$$

We stress that an equilibrium control is not optimal in the sense that it maximizes the value function. However, the control is optimal in the "intuitive" sense that it maximizes the investor's utility given that the investor continues to use the control. Björk et al. (2014) prove neither existence nor uniqueness of the equilibrium control, so there might be several or even no equilibrium controls.

We are now ready to state the key result of this section. The proof is in Appendix A.

Theorem 3.5 (Verification theorem). *Assume there exist a 5 tuple of functions $(U, l_1, l_2, c^*, \pi^*)$ such that the following holds:*

(i) Regularity:

- *U is in $C^{1,2,2}([0,T] \times \mathbb{R} \times \mathbb{R})$.*
- *(c^*, π^*) are admissible controls.*
- *$l_1(t, s, x, y)$ and $l_2(t, T, x, y)$ are defined for $0 \le t \le s \le T$ and $x, y \in \mathbb{R}$, l_1, l_2 are C^1 in t and C^2 in x, y, and l_1 is jointly continuous in t, s.*

(ii) Equilibrium: *The function U solves the pseudo-Bellman equation*

$$
(9) \qquad \partial_t U(t, x, y) = \inf_{c, \pi} \left\{ \begin{bmatrix} -F\left(c, y, \bar{U}(t, x, y)\right) - \mathcal{A}^{c,\pi} U(t, x, y) \\ +\frac{1}{2} \sigma^{c,\pi}(t, x, y)^2 R_x(t, x, y) \\ +\frac{1}{2} \beta(t, y)^2 R_y(t, x, y) \\ +\rho \beta(t, y) \sigma^{c,\pi}(t, x, y) R_{xy}(t, x, y) \end{bmatrix} \right\},
$$

$$
U(T, x, y) = f\left(\omega \delta [\varphi \circ u_2](y, x)\right) ,
$$

where the infimum ranges over all admissible controls, the function F is given by

$$
(10) \qquad F\left(c, y, \bar{U}\right) = \delta \left[f' \circ f^{-1}\right] \left(\bar{U}\right) \cdot \left([\varphi \circ u_1](y, c) - f^{-1}\left(\bar{U}\right) \right) ,
$$

and the functions $\bar{U}$, R_x, R_y, and R_{xy} are given in (iv). The controls (c^*, π^*) realize the infimum in (9).

(iii) Diffusion equations: *For each fixed $0 \le s \le T$ the function l_1 solves the partial differential equation (PDE)*

$$
(11) \qquad \begin{aligned} \partial_t l_1(t, s, x, y) &= -\mathcal{A}^{c^*, \pi^*}(t, s, x, y) l_1(t, s, x, y) , \\ l_1(s, s, x, y) &= u_1(y, c^*(s, x, y)) . \end{aligned}
$$

and the function l_2 solves the PDE

$$
(12) \qquad \begin{aligned} \partial_t l_2(t, T, x, y) &= -\mathcal{A}^{c^*, \pi^*}(t, T, x, y) l_2(t, T, x, y) , \\ l_2(T, T, x, y) &= u_2(y, x) . \end{aligned}
$$

(iv) Remainder functions: *Omitting x, y-dependence, the function $\bar{U}$ is given by*

$$
\bar{U}(t, x) = f\left(\int_t^T \delta e^{-\delta(s-t)} [\varphi \circ l_1](t, s) \, \mathrm{d}s + \omega \delta e^{-\delta(T-t)} [\varphi \circ l_2](t, T) \right),
$$

the remainder term R_x is given by

$$
\begin{aligned}
R_x(t) = &\left[f'' \circ f^{-1}\right]\left(\bar{U}(t)\right) \\
&\cdot \left(\begin{array}{l} \int_t^T \delta e^{-\delta(s-t)} [\varphi' \circ l_1](t, s) \partial_x l_1(t, s) \, \mathrm{d}s \\ +\omega \delta e^{-\delta(T-t)} [\varphi' \circ l_2](t, T) \partial_x l_2(t, T) \end{array} \right)^2 \\
&+ \left[f' \circ f^{-1}\right]\left(\bar{U}(t)\right) \\
&\cdot \left(\begin{array}{l} \int_t^T \delta e^{-\delta(s-t)} [\varphi'' \circ l_1](t, s) (\partial_x l_1(t, s))^2 \, \mathrm{d}s \\ +\omega \delta e^{-\delta(T-t)} [\varphi'' \circ l_2](t, T) (\partial_x l_2(t, T))^2 \end{array} \right) ,
\end{aligned}
$$

and analogously for R_y. The cross-term R_{xy} is defined as

$$R_{xy}(t) = \left[f'' \circ f^{-1} \right] \left(\bar{U}(t) \right)$$

$$\cdot \left(\begin{array}{l} \int_t^T \delta e^{-\delta(s-t)} \left[\varphi' \circ l_1 \right](t,s) \partial_x l_1(t,s)\, \mathrm{d}s \\ + \delta e^{-\delta(T-t)} \left[\varphi' \circ l_2 \right](t,T) \partial_x l_2(t,T) \end{array} \right)$$

$$\cdot \left(\begin{array}{l} \int_t^T \delta e^{-\delta(s-t)} \left[\varphi' \circ l_1 \right](t,s) \partial_y l_1(t,s)\, \mathrm{d}s \\ + \delta e^{-\delta(T-t)} \left[\varphi' \circ l_2 \right](t,T) \partial_y l_2(t,T) \end{array} \right)$$

$$+ \left[f' \circ f^{-1} \right] \left(\bar{U}(t) \right)$$

$$\cdot \left(\begin{array}{l} \int_t^T \delta e^{-\delta(s-t)} \left[\varphi'' \circ l_1 \right](t,s) \partial_y l_1(t,s)\, \partial_x l_1(t,s)\, \mathrm{d}s \\ + \delta e^{-\delta(T-t)} \left[\varphi'' \circ l_2 \right](t,T) \partial_y l_2(t,T)\, \partial_x l_2(t,T) \end{array} \right).$$

Then the following holds:

(i) *The controls (c^*, π^*) are an equilibrium control for the function $V^{c,\pi}$ defined in (6).*

(ii) *The corresponding equilibrium value function V^{c^*, π^*} is given by*

$$V^{c^*, \pi^*}(t,x,y) = \bar{U}(t,x,y) = U(t,x,y).$$

(iii) *For $0 \le t \le s \le T$, we have*

$$m_1^{c^*, \pi^*}(t,s,x,y) = l_1(t,s,x,y)\ ,$$
$$m_2^{c^*, \pi^*}(t,T,x,y) = l_2(t,T,x,y)\ .$$

Remark 3.6. Some brief remarks on the pseudo-Bellman equation that allows for a comparison with the standard examples.

(i) It may appear unusual to have the $\bar{U}$ as an auxiliary function in the aggregator and indeed one could state the theorem with $\bar{U}$ replaced by U. However, this would lead to a highly nonlinear PDE that would require a more sophisticated mathematical treatment which is beyond the scope of this paper.

(ii) In the pseudo-Bellman equation, the terms have the following meaning:

- The aggregator is given by $F\left(c, y, \bar{U}(t,x,y) \right)$.
- The market dynamics are represented by the operator $\mathcal{A}^{c,\pi} U(t,x,y)$.
- The twisting with f and φ introduces the remainder terms $R_x(t,x,y)$, $R_y(t,x,y)$, and $R_{xy}(t,x,y)$.

(iii) The aggregator can be decomposed in two terms:

- The first term $\delta \left[f' \circ f^{-1} \right] \left(\bar{U}(t,x,y) \right) \left[\varphi \circ u_1 \right](y,c)$ is multiplicative in the form: function of $\bar{U}$ times function of c. The twisting yields to a additive perturbation of the standard Bellman equation with explicit remainder terms R_x, R_y, R_{xy}.
- The second term $-\delta \left[f' \circ f^{-1} \right] \left(\bar{U}(t,x,y) \right) f^{-1} \left(\bar{U}(t,x,y) \right)$ is due to the discounting $\delta e^{-\delta(s-t)}$.

(iv) From the above analysis we can expect the same structure of the Bellman equation in higher dimensions, i.e. if we add further diffusion processes.

4 Examples for Incomplete Markets

In this section, we present two non-trivial examples of the framework in incomplete markets. We consider an investor making decisions concerning consumption and investment in the incomplete market model formalized by

$$\mathrm{d}B_t = rB_t \, \mathrm{d}t \ ,$$
$$\mathrm{d}S_t = S_t \left[(r + \lambda(t, Y_t)) \, \mathrm{d}t + \sigma_S(t, Y_t) \, \mathrm{d}W_t \right] \ ,$$
$$\mathrm{d}Y_t = \mu_Y(t, Y_t) \, \mathrm{d}t + \sigma_Y(t, Y_t) \left(\rho \, \mathrm{d}W_t + \sqrt{1 - \rho^2} \, \mathrm{d}\bar{W}_t \right) \ ,$$

where $\lambda, \sigma_S, \mu_Y, \sigma_Y$ are regular functions of (t, Y_t), and W and $\bar{W}$ are two independent Brownian motions. The processes B and S represent price processes of a traded bond and stock whereas Y is an additional non-traded state process driving the coefficients of S. The parameter ρ models the level of correlation between S and Y.

We consider an investor investing the proportion π of his wealth in the stock S and the proportion $(1 - \pi)$ in the bond B and consuming at rate c. The wealth of the investor evolves according to the dynamics

$$(13) \qquad \mathrm{d}X_t^{c,\pi} = X_t^{c,\pi} \left(r + \pi(t, X_t^{c,\pi}, Y_t)\lambda(t, Y_t) \right) \mathrm{d}t - c(t, X_t^{c,\pi}, Y_t) \, \mathrm{d}t$$
$$+ X_t^{c,\pi} \pi(t, X_t^{c,\pi}, Y_t) \sigma_S(t, Y_t) \, \mathrm{d}W_t \ .$$

In the notation from the previous section, we have

$$\alpha(t, y) = \mu_Y(t, y) \ ,$$
$$\beta(t, y) = \sigma_Y(t, y) \ ,$$
$$\mu^{c,\pi}(t, x, y) = x \left(r + \pi(t, x, y)\lambda(t, y) \right) - c(t, x, y) \ ,$$
$$\sigma^{c,\pi}(t, x, y) = x\pi(t, x, y)\sigma_S(t, y) \ .$$

Consumption-investment problems with wealth dynamics given by (13) have been studied with various specifications of $\lambda, \sigma_S, \mu_Y, \sigma_Y$, and ρ by a number of authors. They include Wachther (2002) who works in a complete market setting with constant asset volatility and stochastic excess return linear in Y which is modeled by an Ornstein-Uhlenbeck process, i.e. μ_Y affine in Y and σ_Y constant; Chacko and Viceira (2005) who works with constant excess return and stochastic volatility inverse in the square-root of Y which is modelled by a certain square root process, such that μ_Y is affine in Y and σ_Y is linear in its square-root; Liu (2007) who works with an excess return linear in Y and stochastic volatility equal to the square-root of Y which is modeled by a square root process similar to the one used by Chacko and Viceira (2005). Kraft et al. (2013) consider the model in its generality. In order to avoid complicating issues in connection with stochastic interest rates, see Korn and Kraft (2002), it is important that Y governs the coefficients of S only and not the interest rate. Musiela and Zariphopoulou (2010) refer to the model as a Markovian single stochastic factor model.

Below, we consider the optimization problem in Section 2 for two different choices of the utility functions, namely power utility and exponential utility. This leads to certain constraints on the coefficient functions $\lambda, \sigma_S, \mu_Y, \sigma_Y$.

4.1 Power Utility

To state our result we need the following assumptions on the coefficients driving the financial market dynamics.

Hypothesis 4.1.

(i) Utility functions: let $u_1(y, \xi) = u_2(y, \xi) = y^{\kappa(1-\gamma)}\xi^{1-\gamma}$ for fixed $\kappa \in \mathbb{R}$ and $\gamma \in \mathbb{R}^+, \gamma \neq 1$.

(ii) Elasticity of Inter-temporal Substitution and twisting: let $\varphi(\xi) = \xi^{\frac{1}{\theta}}$ and $f(\xi) = \xi^\theta$ where $\theta = \frac{1-\gamma}{1-\phi}$ for fixed $\phi \in \mathbb{R}^+, \phi \neq 1$.

(iii) Market price of risk: there is a function $h : [0, T] \to \mathbb{R}$ such that $\lambda(t, y) = h(t)\sigma_S(t, y)$.

(iv) Dynamics of Y: there are functions $\alpha, \beta : [0, T] \to \mathbb{R}$ such that $\mu_Y(t, y) = \alpha(t)y$ and $\sigma_Y(t, y) = \beta(t)y$.

To avoid discussions about regularity we assume that all functions are smooth in their arguments. These assumptions ensure that the Verification Theorem is satisfied.

Theorem 4.2 (Equilibrium controls). *Define the function $g : [0, T] \to \mathbb{R}$ by*

$$g(t) = \delta^\theta \left(\int_t^T e^{-\int_t^s \frac{1}{\theta\phi}\tau(v)\,\mathrm{d}v}\,\mathrm{d}s + e^{-\int_t^T \frac{1}{\theta\phi}\tau(v)\,\mathrm{d}v} \omega^{\frac{1}{\phi}} \right)^{\phi\theta} ,$$

where

$$\tau(t) = \left(1 - \frac{1}{\gamma}\right)\left[\rho(1-\gamma)h(t)\beta(t)\kappa + \tfrac{1}{2}h(t)^2 + \tfrac{1}{2}\rho^2\beta(t)^2(1-\gamma)^2\kappa^2\right]$$
$$+ \delta\theta - (1-\gamma)\left[r + \alpha(t)\kappa + \tfrac{1}{2}(\beta(t))^2\kappa((1-\gamma)\kappa - 1)\right] .$$

Also, define the functions $\eta_1, \eta_2 : [0, T]^2 \to \mathbb{R}$ by

$$\eta_1(t, s) = \delta^{\frac{1-\gamma}{\phi}} g(s)^{-\frac{1-\gamma}{\theta\phi}} e^{-\int_t^s \psi(v)\,\mathrm{d}v} ,$$
$$\eta_2(t, T) = e^{-\int_t^T \psi(v)\,\mathrm{d}v} ,$$

where

$$\psi(t) = \tau(t) - \delta\theta + (1-\gamma)g(t)^{-\frac{1}{\phi\theta}}\delta^{\frac{1}{\phi}} .$$

Finally, set

$$c^*(t, x) = \delta^{\frac{1}{\phi}} g(t)^{-\frac{1}{\theta\phi}} x ,$$
$$\pi^*(t, y) = \frac{h(t) + \rho\beta(t)(1-\gamma)\kappa}{\sigma_S(t, y)\gamma} ,$$
$$l_1(t, s, x, y) = \eta_1(t, s)x^{1-\gamma}y^{\kappa(1-\gamma)} ,$$
$$l_2(t, T, x, y) = \eta_2(t, T)x^{1-\gamma}y^{\kappa(1-\gamma)} ,$$
$$U(t, x, y) = x^{1-\gamma}y^{\kappa(1-\gamma)}g(t) ,$$

where we note that c^* is independent of y and π^* is independent of x. Then Theorem 3.5 is satisfied for $(U, l_1, l_2, c^*, \pi^*)$, i.e. the controls (c^*, π^*) are equilibrium controls and U is the corresponding value function.

We note that the equilibrium controls (c^*, π^*) are admissible, in particular Hypothesis 3.1 is satisfied. The proof is in Appendix B.

We noted in Theorem 4.2 that the equilibrium consumption rate becomes independent of Y. This is due to the specific role and form of Y formalized in Hypothesis 4.1. If we e.g. think of Y as a consumer price index and take $\kappa = -1$, we measure utility of goods bought rather than money spent. In that case, and with homogeneity in the sense of (iii) and (iv), it is reasonable to achieve that the optimal consumption is not a function of prices but of wealth (and time) only. Consumption is just a matter of spreading the spending of wealth over time, while hedging away prices changes to the extent possible, and then taking the non-hedgeable part as it comes.

Lemma 4.3. *Under the above assumptions, the dynamics of the equilibrium consumption can be expressed in terms of the SDE*

$$\frac{\mathrm{d}c^*\left(t, X_t^{c^*,\pi^*}\right)}{c^*\left(t, X_t^{c^*,\pi^*}\right)} = \left(r + \tfrac{1}{\gamma}(h(t) + \rho(1-\gamma)\kappa\beta(t))h(t) - \tfrac{\tau(t)}{\theta\phi}\right)\mathrm{d}t$$
$$+ \tfrac{1}{\gamma}\left(h(t) + \rho(1-\gamma)\kappa\beta(t)\right)\mathrm{d}W_t$$

with τ as above.

The proof is in Appendix B.

Remark 4.4. In the introduction we announced a coincidence with the optimal control arising from recursive utility with Epstein-Zin preferences in a Merton market. This coincidence is obtained by setting $\kappa = 0$ and letting λ and σ_S be independent of y such that the state process Y is taken out of the problem. In this case, the equilibrium consumption and investment specified above are exactly those obtained from recursive utility with Epstein-Zin preferences in the same market. It may not be immediately recognizable from the expressions above. Actually, it is easier to recognize from the pseudo-Bellman equation characterizing the solution. The Bellman equation is specialized to power utility in Appendix B. With power utility and $\kappa = 0$, the aggregator in (10) becomes

$$(14) \qquad F\left(c, U\right) = \delta\theta U\left(\left(\frac{c}{U^{\frac{1}{1-\gamma}}}\right)^{1-\phi} - 1\right).$$

This is immediately recognized as the Epstein-Zin normalized aggregator, see e.g. Kraft et al. (2013). We highlight this from Appendix B because of the special role of the coincidence. Since further, in this case, the remainder functions R_x, R_y, and R_{xy} become zero (see Appendix B), the pseudo-Bellman equation in (9) characterizing the equilibrium value function coincides with the Bellman equation characterizing the value function for recursive utility with Epstein-Zin preferences. Therefore the resulting controls also coincide.

Actually, even including the state process Y in the case where the market price of risk is independent of Y ((iii) and (iv) in Hypothesis 4.1) is a special case that has been commented on by others. Kraft et al. (2013) realize that this is a mathematically tractable case but pay little attention to it. Note however that they only consider the case corresponding to $\kappa = 0$. We think that the case certainly deserves attention, in particular since $\kappa \neq 0$ gives a meaningful interpretation of Y as a non-hedgeable consumer price index.

Note from Lemma 4.3 that the equilibrium consumption rate forms a time-inhomogenous geometric Brownian motion. This form, well-known to arise from recursive utility with Epstein-Zin preferences, in general, and from expected time-additive power utility, in particular, is thus kept under the specifications studied in this section.

4.2 Exponential Utility

We define a one-parameter family of functions

$$\epsilon(t) = \frac{r}{1 + Ce^{rt}} \ ,$$

where C is a constant of our choice. Typically, we choose $C = (r - 1)e^{-rT}$ such that $\epsilon(T) = 1$ and the two utility function below coincide.

To state the result we need the following assumptions on the coefficients driving the financial market dynamics.

Hypothesis 4.5.

(i) Utility functions: let $u_1(y, \xi) = \exp(\gamma\kappa y + \gamma\xi)$ and $u_2(y, \xi) = \exp(\gamma\kappa y + \gamma\epsilon(T)\xi)$ for fixed $\kappa \in \mathbb{R}$ and $\gamma \in \mathbb{R}^-$.

(ii) Elasticity of Inter-temporal Substitution and twisting: let $\varphi(\xi) = \xi^{\frac{1}{\theta}}$ and $f(\xi) = \xi^\theta$.

(iii) Market price of risk: there is $h : [0, T] \to \mathbb{R}$ with $\lambda(t, y) = h(t)\sigma_S(t, y)$.

(iv) Dynamics of Y: there are $\alpha, \beta : [0, T] \to \mathbb{R}$ with $\mu_Y(t, y) = \alpha(t)$ and $\sigma_Y(t, y) = \beta(t)$.

For simplicity we assume that all functions are smooth. These assumptions ensure that the Verification Theorem is satisfied.

Theorem 4.6 (Equilibrium controls)**.** *Define the function $g : [0, T] \to \mathbb{R}$ by*

$$g(t) = \exp\left(e^{-\int_t^T \epsilon(v)\,\mathrm{d}v}\theta\log(\delta\omega) - \int_t^T e^{-\int_t^s \epsilon(v)\,\mathrm{d}v}\tau(s)\,\mathrm{d}s\right) \ ,$$

where

$$\tau(t) = \theta\left(\delta - \epsilon(t)\right) + \theta\epsilon(t)\log\left(\frac{\epsilon(t)}{\delta}\right) + \frac{1}{2}h(t)^2 + \frac{1}{2}\rho^2\beta(t)^2\gamma^2\kappa^2$$

$$- \alpha(t)\gamma\kappa - \frac{1}{2}(\beta(t))^2\gamma^2\kappa^2 + \rho\beta(t)\gamma\kappa h(t) \ .$$

Also, define the functions $\eta_1, \eta_2 : [0, T]^2 \to \mathbb{R}$ by

$$\eta_1(t, s) = \left(\frac{\epsilon(s)}{\delta}\right)^\theta g(s) e^{-\int_t^s \psi(v)\,\mathrm{d}v} \ ,$$

$$\eta_2(t, T) = e^{-\int_t^T \psi(v)\,\mathrm{d}v} \ ,$$

where

$$\psi(t) = \tau(t) + \epsilon(t)\log\left(g(t)\right) - \theta\left(\delta - \epsilon(t)\right) \ .$$

Finally, set

$$c^*(t, x) = \epsilon(t)x + \frac{\theta}{\gamma} \log\left(\frac{\epsilon(t)}{\delta}\right) + \frac{1}{\gamma} \log\left(g\left(t\right)\right) \ ,$$

$$\pi^*(t, x, y) = -\frac{1}{x} \frac{h(t) + \rho\beta\left(t\right)\gamma\kappa}{\sigma_S(t, y)\epsilon(t)\gamma} \ ,$$

$$l_1(t, s, x, y) = \eta_1(t, s) \exp(\gamma\kappa y + \gamma\epsilon(t)x) \ ,$$

$$l_2(t, T, x, y) = \eta_2(t, T) \exp(\gamma\kappa y + \gamma\epsilon(T)x) \ ,$$

$$U(t, x, y) = g\left(t\right) \exp(\gamma\kappa y + \gamma\epsilon(t)x) \ ,$$

where we note that c^* is independent of y, and $x\pi^*$ is independent of x. Then Theorem 3.5 is satisfied for $(U, l_1, l_2, c^*, \pi^*)$, i.e. the controls (c^*, π^*) are equilibrium controls and U is the corresponding value function.

We note that the equilibrium controls (c^*, π^*) are admissible, in particular Hypothesis 3.1 is satisfied. The proof is in Appendix C.

Lemma 4.7. *Under the above assumptions, the dynamics of the equilibrium consumption can be expressed in term of the SDE*

$$dc^*\left(t, X_t^{c^*, \pi^*}\right) = \left(\begin{array}{c} \frac{\theta}{\gamma}(\epsilon(t) - r) + \frac{1}{\gamma}\tau(t) - \frac{\theta}{\gamma}\epsilon(t) \log\left(\frac{\epsilon(t)}{\delta}\right) \\ -\frac{1}{\gamma}(h(t) + \rho\kappa\gamma\beta(t))h(t) \end{array}\right) dt$$
$$- \frac{1}{\gamma}\left(h(t) + \rho\kappa\gamma\beta(t)\right) dW_t$$

with τ and ϵ as above.

The proof is in Appendix C.

Remark 4.8. As for power utility, we obtain a coincidence with known preferences from recursive utility by setting $\kappa = 0$ and making sure that (iii) and (iv) in Hypothesis 4.5 are satisfied such that the remainder functions R_x, R_y, and R_{xy} become zero (for details, see Appendix C). With exponential utility and $\kappa = 0$, the aggregator in (10) specializes to

$$F\left(c, U\right) = \delta\theta U \left(\frac{\exp\left(\frac{\gamma}{\theta}c\right)}{U^{\frac{1}{\theta}}} - 1\right) \ .$$

Up to a constant, this coincides with the aggregator arising from the specification $u(c) = \frac{1}{\gamma}\exp(\gamma c)$ and $g(c) = \frac{\theta}{\gamma}\exp(\frac{\gamma}{\theta}c)$ in Section 6 of Kraft and Seifried (2014) which yields the aggregator

$$f\left(c, U\right) = \delta\theta U \left(\frac{\exp(\frac{\gamma}{\theta}c)}{(\gamma U)^{\frac{1}{\theta}}} - 1\right) \ .$$

We highlight this because it shows that the coincidence with recursive utility goes beyond power utility and Epstein-Zin preferences.

Note from Lemma 4.7 that the equilibrium consumption rate forms a time-inhomogenous Brownian motion with drift. This is what could be expected from the wealth-non-memorability feature of the exponential utility function. From a decision-making point of view, the exponential utility function thereby, again, proves to be a questionable specification of preferences. However, its usability in indifference pricing is still reason enough to show all the results in detail here, parallel with power utility.

Appendix

A   Proof of Verification Theorem

The proof of the verification theorem is described in detail in five lemmas.

Proof of Theorem 3.5. We prove the assertions in reverse order.

Assertion (iii) that $m_i^{c^*,\pi^*} = l_i$ for $i = 1, 2$ is in Lemma A.1.

Assertion (ii) on the characterization of the Value function is split into Corollary A.2 which says that $V^*(t,x,y) = \bar{U}(t,x,y)$ and Lemma A.3 giving $V^*(t,x,y) = U(t,x,y)$.

Finally, assertion (i) that (c^*, π^*) are equilibrium controls is proved in Lemma A.4   □

The first lemma characterizes the functions $m_i^{c^*,\pi^*}$ as PDE solutions.

Lemma A.1. *Under the assumptions of Theorem 3.5 it holds that*

$$m_1^{c^*,\pi^*}(t,s,x,y) = l_1(t,s,x,y) \ ,$$
$$m_2^{c^*,\pi^*}(t,T,x,y) = l_2(t,T,x,y)$$

for any $0 \le t \le s \le T$.

Proof. Fix an admissible control (c,π). By Definition 3.2 there exist functions $\Lambda_1^{c,\pi}(t,s,x,y)$ and $\Lambda_2^{c,\pi}(t,T,x,y)$, defined for $0 \le t \le s \le T$ and $x, y \in \mathbb{R}$, such that

- $\Lambda_1^{c,\pi}, \Lambda_2^{c,\pi}$ are C^1 in t and C^2 in x, y, and $\Lambda_1^{c,\pi}$ is jointly continuous in t, s.

- For each fixed $0 \le s \le T$, $\Lambda_1^{c,\pi}$ solves the PDE

$$(15) \qquad \left. \begin{aligned} \partial_t \Lambda_1^{c,\pi}(t,s,x,y) &= -\mathcal{A}^{c,\pi} \Lambda_1^{c,\pi}(t,s,x,y) \ , \\ \Lambda_1^{c,\pi}(s,s,x,y) &= u_1(y, c(s,x,y)) \ . \end{aligned} \right\}$$

- $\Lambda_2^{c,\pi}$ solves the PDE

$$(16) \qquad \left. \begin{aligned} \partial_t \Lambda_2^{c,\pi}(t,T,x,y) &= -\mathcal{A}^{c,\pi} \Lambda_2^{c,\pi}(t,T,x,y) \ , \\ \Lambda_2^{c,\pi}(T,T,x,y) &= u_2(y,x) \ . \end{aligned} \right\}$$

By the classical Feynman–Kac theorem we have

$$\Lambda_1^{c,\pi}(t,s,x,y) = \mathbb{E}_{t,x,y}[u_1(Y_s, c(s,X_s^{c,\pi},Y_s))] = m_1^{c,\pi}(t,s,x,y) \ , \quad t \le s < T \ ,$$

and

$$\Lambda_2^{c,\pi}(t,T,x,y) = \mathbb{E}_{t,x,y}[u_2(Y_T, X_T^{c,\pi})] = m_2^{c,\pi}(t,T,x,y) \ , \quad s < T \ .$$

Since solutions of the PDEs are unique, we have $\Lambda_i^{c^*,\pi^*} = l_i$ for $i = 1, 2$.   □

We observe an immediate consequence of the proof.

Corollary A.2. *The value function $V^{c,\pi}$ can be written as*

$$(17) \qquad V^{c,\pi}(t,x,y) = f \left(\begin{array}{c} \int_t^T \delta e^{-\delta(s-t)} [\varphi \circ \Lambda_1^{c,\pi}](t,s,x,y) \, \mathrm{d}s \\ + \omega \delta e^{-\delta(T-t)} [\varphi \circ \Lambda_2^{c,\pi}](t,T,x,y) \end{array} \right) \ ,$$

and, in particular, the equilibrium value function satisfies $V^{c^*,\pi^*} = \bar{U}$

Lemma A.3. *The equilibrium value function satisfies $V^{c^*,\pi^*} = U$.*

Proof. Since $\Lambda_1^{c,\pi}, \Lambda_2^{c,\pi}$ are C^1 in t and C^2 in x, y, $\Lambda_1^{c,\pi}$ is jointly continuous in t, s, and φ is in C^2, we get from the representation in (17) that $V^{c,\pi}$ is in $C^{1,2,2}$. Suppressing x, y-dependence in the Λ-functions, we obtain the partial derivatives (for $i = x, y$)

$$\partial_t V^{c,\pi}(t,x,y)$$
$$= -\delta \left[f' \circ f^{-1} \right] (V^{c,\pi}(t,x,y)) \left([\varphi \circ u_1](y, c(t,x,y)) - f^{-1}(V^{c,\pi}(t,x,y)) \right)$$
$$+ \left[f' \circ f^{-1} \right] (V^{c,\pi}(t,x,y)) \left(\begin{array}{l} \int_t^T \delta e^{-\delta(s-t)} [\varphi' \circ \Lambda_1^{c,\pi}](t,s) \, \partial_t \Lambda_1^{c,\pi}(t,s) \, \mathrm{d}s \\ + \omega \delta e^{-\delta(T-t)} [\varphi' \circ \Lambda_2^{c,\pi}](t,T) \, \partial_t \Lambda_2^{c,\pi}(t,T) \end{array} \right),$$

$$\partial_i V^{c,\pi}(t,x,y)$$
$$= \left[f' \circ f^{-1} \right] (V^{c,\pi}(t,x,y)) \left(\begin{array}{l} \int_t^T \delta e^{-\delta(s-t)} [\varphi' \circ \Lambda_1^{c,\pi}](t,s) \, \partial_i \Lambda_1^{c,\pi}(t,s) \, \mathrm{d}s \\ + \omega \delta e^{-\delta(T-t)} [\varphi' \circ \Lambda_2^{c,\pi}](t,T) \, \partial_i \Lambda_2^{c,\pi}(t,T) \end{array} \right),$$

$$\partial_i^2 V^{c,\pi}(t,x,y)$$
$$= \left[f'' \circ f^{-1} \right] (V^{c,\pi}(t,x,y)) \left(\begin{array}{l} \int_t^T \delta e^{-\delta(s-t)} [\varphi' \circ \Lambda_1^{c,\pi}](t,s) \, \partial_i \Lambda_1^{c,\pi}(t,s) \, \mathrm{d}s \\ + \omega \delta e^{-\delta(T-t)} [\varphi' \circ \Lambda_2^{c,\pi}](t,T) \, \partial_i \Lambda_2^{c,\pi}(t,T) \end{array} \right)^2$$
$$+ \left[f' \circ f^{-1} \right] (V^{c,\pi}(t,x,y)) \left(\begin{array}{l} \int_t^T \delta e^{-\delta(s-t)} [\varphi'' \circ \Lambda_1^{c,\pi}](t,s) \, (\partial_i \Lambda_1^{c,\pi}(t,s))^2 \, \mathrm{d}s \\ + \omega \delta e^{-\delta(T-t)} [\varphi'' \circ \Lambda_2^{c,\pi}](t,T) \, (\partial_i \Lambda_2^{c,\pi}(t,T))^2 \end{array} \right)$$
$$+ \left[f' \circ f^{-1} \right] (V^{c,\pi}(t,x,y)) \left(\begin{array}{l} \int_t^T \delta e^{-\delta(s-t)} [\varphi' \circ \Lambda_1^{c,\pi}](t,s) \, \partial_i^2 \Lambda_1^{c,\pi}(t,s) \, \mathrm{d}s \\ + \omega \delta e^{-\delta(T-t)} [\varphi' \circ \Lambda_2^{c,\pi}](t,T) \, \partial_i^2 \Lambda_2^{c,\pi}(t,T) \end{array} \right),$$

$$\partial_{xy} V^{c,\pi}(t,x,y)$$
$$= \left[f' \circ f^{-1} \right] (V^{c,\pi}(t,x,y))$$
$$\cdot \left(\begin{array}{l} \int_t^T \delta e^{-\delta(s-t)} [\varphi'' \circ \Lambda_1^{c,\pi}](t,s) \, \partial_x \Lambda_1^{c,\pi}(t,s) \, \partial_y \Lambda_1^{c,\pi}(t,s) \, \mathrm{d}s \\ + \omega \delta e^{-\delta(T-t)} [\varphi'' \circ \Lambda_2^{c,\pi}](t,T) \, \partial_x \Lambda_2^{c,\pi}(t,T) \, \partial_y \Lambda_2^{c,\pi}(t,T) \end{array} \right)$$
$$+ \left[f'' \circ f^{-1} \right] (V^{c,\pi}(t,x,y)) \left(\begin{array}{l} \int_t^T \delta e^{-\delta(s-t)} [\varphi' \circ \Lambda_1^{c,\pi}](t,s) \, \partial_x \Lambda_1^{c,\pi}(t,s) \, \mathrm{d}s \\ + \omega \delta e^{-\delta(T-t)} [\varphi' \circ \Lambda_2^{c,\pi}](t,T) \, \partial_x \Lambda_2^{c,\pi}(t,T) \end{array} \right)$$
$$\cdot \left(\begin{array}{l} \int_t^T \delta e^{-\delta(s-t)} [\varphi' \circ \Lambda_1^{c,\pi}](t,s) \, \partial_y \Lambda_1^{c,\pi}(t,s) \, \mathrm{d}s \\ + \omega \delta e^{-\delta(T-t)} [\varphi' \circ \Lambda_2^{c,\pi}](t,T) \, \partial_y \Lambda_2^{c,\pi}(t,T) \end{array} \right)$$
$$+ \left[f' \circ f^{-1} \right] (V^{c,\pi}(t,x,y)) \cdot \left(\begin{array}{l} \int_t^T \delta e^{-\delta(s-t)} [\varphi' \circ \Lambda_1^{c,\pi}](t,s) \, \partial_{xy} \Lambda_1^{c,\pi}(t,s) \, \mathrm{d}s \\ + \omega \delta e^{-\delta(T-t)} [\varphi' \circ \Lambda_2^{c,\pi}](t,T) \, \partial_{xy} \Lambda_2^{c,\pi}(t,T) \end{array} \right).$$

Applying (15)–(16), we obtain the PDE

$$(18) \quad \left. \begin{array}{l} \partial_t V^{c,\pi}(t,x,y) = -F(c(t,x,y), y, V^{c,\pi}(t,x,y)) - \mathcal{A}^{c,\pi} V^{c,\pi}(t,x,y) \\ \qquad\qquad + \frac{1}{2} (\sigma^{c,\pi}(t,x,y))^2 R_x^{c,\pi}(t,x,y) + \frac{1}{2} (\beta(t,y))^2 R_y^{c,\pi}(t,x,y) \\ \qquad\qquad + \rho \beta(t,y) \sigma^{c,\pi}(t,x,y) R_{xy}^{c,\pi}(t,x,y), \\ V^{c,\pi}(T,x,y) = f(\omega \delta [\varphi \circ u_2](y,x)), \end{array} \right\}$$

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where F is given by (10) and (for $i = x, y$)

$$
\begin{aligned}
&R_i^{c,\pi}(t, x, y) \\
&= \left[f'' \circ f^{-1} \right] (V^{c,\pi}(t, x, y)) \left(\begin{array}{l} \int_t^T \delta e^{-\delta(s-t)} \left[\varphi' \circ \Lambda_1^{c,\pi} \right] (t, s) \, \partial_i \Lambda_1^{c,\pi}(t, s) \, \mathrm{d}s \\ +\omega \delta e^{-\delta(T-t)} \left[\varphi' \circ \Lambda_2^{c,\pi} \right] (t, T) \, \partial_i \Lambda_2^{c,\pi}(t, T) \end{array} \right)^2 \\
&\quad + \left[f' \circ f^{-1} \right] (V^{c,\pi}(t, x, y)) \\
&\quad \cdot \left(\begin{array}{l} \int_t^T \delta e^{-\delta(s-t)} \left[\varphi'' \circ \Lambda_1^{c,\pi} \right] (t, s) \left(\partial_i \Lambda_1^{c,\pi}(t, s) \right)^2 \mathrm{d}s \\ +\omega \delta e^{-\delta(T-t)} \left[\varphi'' \circ \Lambda_2^{c,\pi} \right] (t, T) \left(\partial_i \Lambda_2^{c,\pi}(t, T) \right)^2 \end{array} \right) ,
\end{aligned}
$$

$$
\begin{aligned}
&R_{xy}^{c,\pi}(t, x, y) \\
&= \left[f' \circ f^{-1} \right] (V^{c,\pi}(t, x, y)) \\
&\quad \cdot \left(\begin{array}{l} \int_t^T \delta e^{-\delta(s-t)} \left[\varphi'' \circ \Lambda_1^{c,\pi} \right] (t, s) \, \partial_x \Lambda_1^{c,\pi}(t, s) \, \partial_y \Lambda_1^{c,\pi}(t, s) \, \mathrm{d}s \\ +\omega \delta e^{-\delta(T-t)} \left[\varphi'' \circ \Lambda_2^{c,\pi} \right] (t, T) \, \partial_x \Lambda_2^{c,\pi}(t, T) \, \partial_y \Lambda_2^{c,\pi}(t, T) \end{array} \right) \\
&\quad + \left[f'' \circ f^{-1} \right] (V^{c,\pi}(t, x, y)) \\
&\quad \cdot \left(\begin{array}{l} \int_t^T \delta e^{-\delta(s-t)} \left[\varphi' \circ \Lambda_1^{c,\pi} \right] (t, s) \, \partial_x \Lambda_1^{c,\pi}(t, s) \, \mathrm{d}s \\ +\omega \delta e^{-\delta(T-t)} \left[\varphi' \circ \Lambda_2^{c,\pi} \right] (t, T) \, \partial_x \Lambda_2^{c,\pi}(t, T) \end{array} \right) \\
&\quad \cdot \left(\begin{array}{l} \int_t^T \delta e^{-\delta(s-t)} \left[\varphi' \circ \Lambda_1^{c,\pi} \right] (t, s) \, \partial_y \Lambda_1^{c,\pi}(t, s) \, \mathrm{d}s \\ +\omega \delta e^{-\delta(T-t)} \left[\varphi' \circ \Lambda_2^{c,\pi} \right] (t, T) \, \partial_y \Lambda_2^{c,\pi}(t, T) \end{array} \right) .
\end{aligned}
$$

To establish the relation $U = V^{c^*, \pi^*}$, we recall that $\Lambda_i^{c^*, \pi^*} = l_i$, $i = 1, 2$, and $V^{c^*, \pi^*} = \bar{U}$. This implies that $R_x^{c^*, \pi^*} = R_x$, $R_y^{c^*, \pi^*} = R_y$, and $R_{xy}^{c^*, \pi^*} = R_{xy}$, where R_x, R_y, and R_{xy} are given in the theorem. Thus, V^{c^*, π^*} solves the PDE

$$
\left.
\begin{aligned}
\partial_t V^{c^*, \pi^*}(t, x, y) &= -F \left(c^*(t, x, y), y, \bar{U}(t, x, y) \right) - \mathcal{A}^{c^*, \pi^*} V^{c^*, \pi^*}(t, x, y) \\
&\quad + \tfrac{1}{2} \sigma^{c^*, \pi^*}(t, x, y)^2 R_x(t, x, y) + \tfrac{1}{2} \beta(t, y)^2 R_y(t, x, y) \\
&\quad + \rho \beta(t, y) \sigma^{c^*, \pi^*}(t, x, y) R_{xy}(t, x, y) , \\
V^{c^*, \pi^*}(T, x, y) &= f \left(\omega \delta \left[\varphi \circ u_2 \right](y, x) \right) .
\end{aligned}
\right\}
$$

From the Bellman equation, we know that U solves the PDE

$$
\left.
\begin{aligned}
\partial_t U(t, x, y) &= -F \left(c^*(t, x, y), y, \bar{U}(t, x, y) \right) - \mathcal{A}^{c^*, \pi^*} U(t, x, y) \\
&\quad + \frac{1}{2} \sigma^{c^*, \pi^*}(t, x, y)^2 R_x(t, x, y) + \frac{1}{2} \beta(t, y)^2 R_y(t, x, y) \\
&\quad + \rho \beta(t, y) \sigma^{c^*, \pi^*}(t, x, y) R_{xy}(t, x, y) , \\
U(T, x, y) &= f \left(\omega \delta \left[\varphi \circ u_2 \right](y, x) \right) .
\end{aligned}
\right\}
$$

Altogether, the difference $U - V^{c^*, \pi^*}$ solves the PDE

$$
\left.
\begin{aligned}
\partial_t \left(U - V^{c^*, \pi^*} \right)(t, x, y) &= -\mathcal{A}^{c^*, \pi^*} \left(U - V^{c^*, \pi^*} \right)(t, x, y) , \\
\left(U - V^{c^*, \pi^*} \right)(T, x, y) &= 0 .
\end{aligned}
\right\}
$$

Hence, we must have $U = V^{c^*, \pi^*}$. $\qquad\square$

Finally, we show that the controls c^* and π^* are indeed equilibrium controls.

Lemma A.4. *The pair (c^*, π^*) is an equilibrium control.*

Proof. We fix an admissible control $(\bar{c}, \bar{\pi})$, a (small) real number $h > 0$, and an initial point $(u, x, y) \in [0, T] \times \mathbb{R}^2$. We then define the control $\left(c^h, \pi^h\right)$ by

$$\left(c^h, \pi^h\right)(t, \bar{x}, \bar{y}) = \begin{cases} (\bar{c}, \bar{\pi})(t, \bar{x}, \bar{y}) \,, & u \le t < u + h, \bar{x}, \bar{y} \in \mathbb{R} \,, \\ (c^*, \pi^*)(t, \bar{x}, \bar{y}) \,, & u + h \le t \le T, \bar{x}, \bar{y} \in \mathbb{R} \,. \end{cases}$$

Below, we write $V^h = V^{c^h, \pi^h}$, $\Lambda_i^h = \Lambda_i^{c^h, \pi^h}$, and $X^h = X^{c^h, \pi^h}$.

To prove that (c^*, π^*) is an equilibrium control in the sense of Definition 3.4, we need to verify that condition (8) is satisfied. Recall that $V^{c^*, \pi^*} = U$. Hence, Equation (8) reads

$$\liminf_{h \to 0} \frac{U(u, x, y) - V^h(u, x, y)}{h} \ge 0 \,.$$

By construction, we have $V^h(t, x, y) = U(t, x, y) = \bar{U}(t, x, y)$ for $t \ge u + h$. Thus, applying Taylor's formula for fixed x, y, we get that

$$\frac{U(u, x, y) - V^h(u, x, y)}{h}$$
$$= \frac{U(u, x, y) - U(u + h, x, y) - V^h(u, x, y) + V^h(u + h, x, y)}{h}$$
$$= -U_t(u, x, y) + V_t^h(u, x, y) + o(h) \,.$$

Hence, what we need to show is that

$$\liminf_{h \to 0} \left[-U_t(u, x, y) + V_t^h(u, x, y) \right] \ge 0 \,.$$

By (18) and the Bellman equation, we have

$$- U_t(u, x, y) + V_t^h(u, x, y)$$
$$\ge F\left(\bar{c}(u, x, y), y, \bar{U}(u, x, y)\right) - F\left(\bar{c}(u, x, y), y, V^h(u, x, y)\right)$$
$$\quad + \mathcal{A}^{\bar{c}, \bar{\pi}}\left(U(u, x, y) - V^h(u, x, y)\right)$$
$$\quad + \frac{1}{2}(\bar{\sigma}(u, x, y))^2 \left(R_x^h(u, x, y) - R_x(u, x, y)\right)$$
$$\quad + \frac{1}{2}(\beta(u, y))^2 \left(R_y^h(u, x, y) - R_y(u, x, y)\right)$$
$$\quad + \rho\beta(u, y)\bar{\sigma}(u, x, y)\left(R_{xy}^h(u, x, y) - R_{xy}(u, x, y)\right) \,.$$

Hence, it suffices to show that for $h \to 0$

$$(19) \qquad F\left(\bar{c}(u, x, y), y, V^h(u, x, y)\right) \to F\left(\bar{c}(u, x, y), y, \bar{U}(u, x, y)\right) \,,$$

$$(20) \qquad \qquad \mathcal{A}^{\bar{c}, \bar{\pi}} V^h(u, x, y) \to \mathcal{A}^{\bar{c}, \bar{\pi}} U(u, x, y) \,,$$

$$(21) \qquad \qquad R_x^h(u, x, y) \to R_x(u, x, y) \,,$$

$$(22) \qquad \qquad R_y^h(u, x, y) \to R_y(u, x, y) \,,$$

$$(23) \qquad \qquad R_{xy}^h(u, x, y) \to R_{xy}(u, x, y) \,.$$

Since V^h and $\bar{U} = U$ are continuous in the first argument, we note that

$$V^h(u, x, y) = V^h(u, x, y) - V^h(u + h, x, y) + \bar{U}(u + h, x, y)$$
$$\to 0 + \bar{U}(u, x, y) \quad \text{as} \quad h \to 0 .$$

By assumption f and φ are in C^2 and f^{-1} is in C^0. Hence, F is continuous, and (19) follows immediately. Furthermore, since V^h and U are in $C^{1,2,2}$, we get (20):

$$\mathcal{A}^{\bar{c},\bar{\pi}} V^h(u, x, y) = \mathcal{A}^{\bar{c},\bar{\pi}} V^h(u, x, y) - \mathcal{A}^{\bar{c},\bar{\pi}} V^h(u + h, x, y)$$
$$+ \mathcal{A}^{\bar{c},\bar{\pi}} U(u + h, x, y)$$
$$\to 0 + \mathcal{A}^{\bar{c},\bar{\pi}} U(u, x, y) \quad \text{as} \quad h \to 0 .$$

Finally, $f'' \circ f^{-1}$ is continuous, so for (21) to hold, it suffices to show that

$$\int_u^T \delta e^{-\delta(v-u)} \left[\varphi' \circ \Lambda_1^h \right] (u, v, x, y) \, \partial_x \Lambda_1^h(u, v, x, y) \, dv$$
$$+ \omega \delta e^{-\delta(T-u)} \left[\varphi' \circ \Lambda_2^h \right] (u, T, x, y) \, \partial_x \Lambda_2^h(u, T, x, y)$$
$$\to \int_u^T \delta e^{-\delta(v-u)} \left[\varphi' \circ l_1 \right] (u, v, x, y) \, \partial_x l_1(u, v, x, y) \, dv$$
$$+ \omega \delta e^{-\delta(T-u)} \left[\varphi' \circ l_2 \right] (u, T, x, y) \, \partial_x l_2(u, T, x, y) \quad \text{as} \quad h \to 0 ,$$

and

$$\int_u^T \delta e^{-\delta(v-u)} \left[\varphi'' \circ \Lambda_1^h \right] (u, v, x, y) \left(\partial_x \Lambda_1^h(u, v, x, y) \right)^2 \, dv$$
$$+ \omega \delta e^{-\delta(T-u)} \left[\varphi'' \circ \Lambda_2^h \right] (u, T, x, y) \left(\partial_x \Lambda_2^h(u, T, x, y) \right)^2$$
$$\to \int_u^T \delta e^{-\delta(v-u)} \left[\varphi'' \circ l_1 \right] (u, v, x, y) \left(\partial_x l_1(u, v, x, y) \right)^2 \, dv$$
$$+ \omega \delta e^{-\delta(T-u)} \left[\varphi'' \circ l_2 \right] (u, T, x, y) \left(\partial_x l_2(u, T, x, y) \right)^2 \quad \text{as} \quad h \to 0 .$$

This is ensured by the fact that φ is in C^2 and Λ_i^h and l_i are in $C^{1,2,2}$. To see this, realize that $\Lambda_i^h(t, s, x, y) = l_i(t, s, x, y)$ for $s \geq t \geq u + h$ by construction and write

$$\int_u^T \delta e^{-\delta(v-u)} \left[\varphi' \circ \Lambda_1^h \right] (u, v, x, y) \, \partial_x \Lambda_1^h(u, v, x, y) \, dv$$
$$= \int_u^{u+h} \delta e^{-\delta(v-u)} \left[\varphi' \circ \Lambda_1^h \right] (u, v, x, y) \, \partial_x \Lambda_1^h(u, v, x, y) \, dv$$
$$+ \int_{u+h}^T \delta e^{-\delta(v-u)} \left[\varphi' \circ l_1 \right] (u + h, v, x, y) \, \partial_x l_1(u + h, v, x, y) \, dv$$
$$+ \int_{u+h}^T \delta e^{-\delta(v-u)} \left(\begin{array}{c} \left[\varphi' \circ \Lambda_1^h \right] (u, v, x, y) \, \partial_x \Lambda_1^h(u, v, x, y) \\ - \left[\varphi' \circ \Lambda_1^h \right] (u + h, v, x, y) \, \partial_x \Lambda_1^h(u + h, v, x, y) \end{array} \right) dv$$
$$\to 0 + \int_u^T \delta e^{-\delta(v-u)} \left[\varphi' \circ l_1 \right] (u, v, x, y) \, \partial_x l_1(u, v, x, y) \, dv + 0 \quad \text{as} \quad h \to 0 .$$

To complete the proof, we note that (22)–(23) follow from the same arguments as (21). $\qquad \square$

B   Details for the Power Utility Example

Proof of Theorem 4.2. We proceed in 3 steps.

  1. Compute infinitesimal generator and aggregator. With the assumptions on the market dynamics, we get the infinitesimal generator

$$
\begin{aligned}
\mathcal{A}^{c,\pi} &= \left[x\left(r + \pi(t,x,y)\lambda(t,y) - c(t,x,y)\right)\right]\partial_x + \alpha\left(t,y\right)\partial_y \\
&\quad + \tfrac{1}{2}\left(x\pi(t,x,y)\sigma_S(t,y)\right)^2\partial_x^2 + \tfrac{1}{2}\left(\beta\left(t,y\right)\right)^2\partial_y^2 \\
&\quad + \rho\alpha\left(t,y\right)x\pi(t,x,y)\sigma_S(t,y)\partial_{xy} \\
&= \left[x\left(r + \pi(t,x,y)h(t)\sigma_S(t,y)\right) - c(t,x,y)\right]\partial_x + \alpha\left(t\right)y\partial_y \\
&\quad + \tfrac{1}{2}\left(x\pi(t,x,y)\sigma_S(t,y)\right)^2\partial_x^2 + \tfrac{1}{2}\left(\beta\left(t\right)y\right)^2\partial_y^2 \\
&\quad + \rho\beta\left(t\right)yx\pi(t,x,y)\sigma_S(t,y)\partial_{xy}\ .
\end{aligned}
$$

Also, with power utility we get the aggregator

$$
\begin{aligned}
F\left(c,y,U\right) &= \delta\theta\left(U^{\frac{1}{\theta}}\right)^{\theta-1}\left(y^{\frac{\kappa(1-\gamma)}{\theta}}c^{\frac{1-\gamma}{\theta}} - U^{\frac{1}{\theta}}\right) \\
&= \delta\theta U\left(\left(\frac{y^\kappa c}{U^{\frac{1}{1-\gamma}}}\right)^{1-\phi} - 1\right)\ .
\end{aligned}
$$

Note that for $\kappa = 0$ we get the classical normalized aggregator in (14) arising in recursive utility with Epstein-Zin preferences, see also the comments in Remark 4.4.

  2. Verify that (c^*,π^*) are equilibrium controls and U solves the Bellman equation. From Theorem 4.2, we have

(24)
$$
l_i(t,s,x,y) = \eta_i(t,s)x^{1-\gamma}y^{\kappa(1-\gamma)}\ ,\quad i = 1,2\ ,
$$

so the remainder terms vanish, i.e. $R_x = R_y = R_{xy} = 0$. We also have

$$
U(t,x,y) = x^{1-\gamma}y^{\kappa(1-\gamma)}g\left(t\right)\ .
$$

Plugging this and $R_x = R_y = R_{xy} = 0$ into the Bellman equation that U must solve, i.e. Equation (9), and dividing by $x^{1-\gamma}y^{\kappa(1-\gamma)}$, we obtain

(25)
$$
\left.\begin{aligned}
g'(t) &= \inf_{c,\pi}\left[\begin{array}{c}
-\delta\theta\left(g\left(t\right)\right)^{1-\frac{1}{\theta}}\left(\left(\frac{c}{x}\right)^{\frac{1-\gamma}{\theta}} - g\left(t\right)^{\frac{1}{\theta}}\right) \\
-\left(\left(r + \pi h(t)\sigma_S(t,y)\right) - \frac{c}{x}\right)\left(1-\gamma\right)g(t) \\
-\tfrac{1}{2}\left(\pi\sigma_S(t,y)\right)^2\left(1-\gamma\right)\left(-\gamma\right)g(t) \\
-\alpha\left(t\right)\left(1-\gamma\right)\kappa g(t) \\
-\tfrac{1}{2}\left(\beta\left(t\right)\right)^2\left(1-\gamma\right)\kappa\left(\left(1-\gamma\right)\kappa - 1\right)g(t) \\
-\rho\beta\left(t\right)\pi\sigma_S(t,y)\left(1-\gamma\right)^2\kappa g(t)
\end{array}\right], \\
g\left(T\right) &= \left(\delta\omega\right)^\theta\ .
\end{aligned}\right\}
$$

The first order conditions for c,π read

$$
\begin{aligned}
0 &= -\delta\theta g\left(t\right)^{1-\frac{1}{\theta}}\frac{1-\gamma}{\theta}c^{\frac{1-\gamma}{\theta}-1}\left(\frac{1}{x}\right)^{\frac{1-\gamma}{\theta}} + \frac{1}{x}(1-\gamma)g(t)\ , \\
0 &= -h(t)\sigma_S(t,y)(1-\gamma)g(t) - \pi\left(\sigma_S(t,y)\right)^2(1-\gamma)(-\gamma)g(t) \\
&\quad - \rho\beta\left(t\right)\sigma_S(t,y)(1-\gamma)^2\kappa g(t)\ .
\end{aligned}
$$

These are satisfied by the controls $c^*(t, x, y)$ and $\pi^*(t, x, y)$ from Theorem 4.2. Plugging the controls into (25), we obtain

$$\left.\begin{aligned} g'(t) &= -\theta\phi \left(g\left(t\right)\right)^{1-\frac{1}{\theta\phi}} \delta^{\frac{1}{\phi}} + \tau(t)g(t) \ , \\ g\left(T\right) &= (\delta\omega)^{\theta} \ , \end{aligned}\right\}$$

where

$$\tau(t) = \left(1 - \frac{1}{\gamma}\right) \left[\rho(1-\gamma)h(t)\beta\left(t\right)\kappa + \frac{1}{2}h(t)^2 + \frac{1}{2}\rho^2\beta\left(t\right)^2 (1-\gamma)^2\kappa^2\right]$$
$$+ \ \delta\theta - (1-\gamma)\left[r + \alpha\left(t\right)\kappa + \frac{1}{2}\left(\beta\left(t\right)\right)^2 \kappa((1-\gamma)\kappa - 1)\right] \ .$$

This ordinary differential equation (ODE) is obviously solved by the function g from Theorem 4.2. Hence, the function U from Theorem 4.2 solves the necessary Bellman equation.

3. Verify that l_1, l_2 solve diffusion equations. Plugging (24) and the controls (c^*, π^*) into the diffusion equations that l_1 and l_2 must solve, i.e. Equations (11)–(12), and dividing by $x^{1-\gamma}y^{\kappa(1-\gamma)}$, we obtain

$$\left.\begin{aligned} \partial_t\eta_1(t, s) &= \psi(t)\eta_1(t, s) \ , \\ \eta_1(s, s) &= \delta^{\frac{1-\gamma}{\phi}} g\left(s\right)^{-\frac{1-\gamma}{\theta\phi}} \ , \end{aligned}\right\}$$

and

$$\left.\begin{aligned} \partial_t\eta_2(t, T) &= \psi(t)\eta_2(t, T) \ , \\ \eta_2(T, T) &= 1 \ , \end{aligned}\right\}$$

where

$$\psi(t) = \tau(t) - \delta\theta + (1-\gamma)g\left(t\right)^{-\frac{1}{\phi\theta}} \delta^{\frac{1}{\phi}} \ .$$

These ODE's are obviously solved by the function η_1, η_2 from Theorem 4.2. Hence, l_1, l_2 from Theorem 4.2 solve the necessary diffusion equations. $\qquad\square$

Proof of Lemma 4.3. The idea is to apply Itô's Lemma. Recall that

$$c^*(t, x) = \delta^{\frac{1}{\phi}} g(t)^{-\frac{1}{\phi\theta}} x \ ,$$

where g satisfies the ODE

$$g'(t) = g(t)\left[-\theta\phi\delta^{\frac{1}{\phi}}g(t)^{1-\frac{1}{\phi\theta}} + \tau(t)g(t)\right] .$$

We get the partial derivatives

$$\partial_t c^*(t, x) = \delta^{\frac{1}{\phi}}\left(-\frac{1}{\theta\phi}\right) g(t)^{-\frac{1}{\phi\theta}-1} g'(t)x$$
$$= -\frac{\delta^{\frac{1}{\phi}}}{\theta\phi} g(t)^{-\frac{1}{\phi\theta}-1}\left(-\theta\phi\delta^{\frac{1}{\phi}}g(t)^{1-\frac{1}{\phi\theta}} + \tau(t)g(t)\right)x$$
$$= \frac{c^*(t, x)^2}{x} - \frac{\tau(t)}{\theta\phi}c^*(t, x) \ ,$$
$$\partial_x c^*(t, x) = \delta^{\frac{1}{\phi}}g(t)^{-\frac{1}{\phi\theta}}$$
$$= \frac{c^*(t, x)}{x} \ ,$$
$$\partial_x^2 c^*(t, x) = 0 \ .$$

The drift and diffusion coefficient under the equilibrium controls are given as

$$
\begin{aligned}
\mu^*(t,x) &= x[r + \pi^*(t,y)\lambda(t,y)] - c^*(t,x) \\
&= x\left[r + \tfrac{1}{\gamma}\left(h(t) + \rho(1-\gamma)\kappa\beta(t)\right)h(t)\right] - c^*(t,x)\ , \\
\sigma^*(t,x) &= x\pi^*(t,y)\sigma_S(t,y) \\
&= x\tfrac{1}{\gamma}\left(h(t) + \rho(1-\gamma)\kappa\beta(t)\right)\ .
\end{aligned}
$$

Applying Itô's Lemma, we obtain

$$
\begin{aligned}
\mathrm{d}c^*\left(t,X_t^{c^*,\pi^*}\right) &= \left(\partial_t c^*(t,X_t^{c^*,\pi^*}) + \mu^*\left(t,X_t^{c^*,\pi^*}\right)\partial_x c^*\left(t,X_t^{c^*,\pi^*}\right)\right)\mathrm{d}t \\
&\quad + \sigma^*\left(t,X_t^{c^*,\pi^*}\right)\partial_x c^*\left(t,X_t^{c^*,\pi^*}\right)\mathrm{d}W_t \\
&= \begin{bmatrix} \frac{c^*(t,X_t^{c^*,\pi^*})^2}{X_t^{c^*,\pi^*}} - \frac{\tau(t)}{\theta\phi}c^*(t,X_t^{c^*,\pi^*}) \\ c^*\left(t,X_t^{c^*,\pi^*}\right)\left[r + \tfrac{1}{\gamma}(h(t) + \rho(1-\gamma)\kappa\beta(t))h(t)\right] \\ -\frac{c^*(t,X_t^{c^*,\pi^*})^2}{X_t^{c^*,\pi^*}} \end{bmatrix}\mathrm{d}t \\
&\quad + c^*(t,X_t^{c^*,\pi^*})\tfrac{1}{\gamma}(h(t) + \rho(1-\gamma)\kappa\beta(t))\,\mathrm{d}W_t\ ,
\end{aligned}
$$

which yields the assertion. $\qquad\square$

C Details for the Exponential Utility Example

Proof of Theorem 4.6. We proceed in 3 steps.

1. Compute infinitesimal generator and aggregator. With the assumptions on the market dynamics, we get the infinitesimal generator

$$
\begin{aligned}
\mathcal{A}^{c,\pi} &= \left[x\left(r + \pi(t,x,y)\lambda(t,y) - c(t,x,y)\right)\right]\partial_x + \alpha\left(t,y\right)\partial_y \\
&\quad + \tfrac{1}{2}\left(x\pi(t,x,y)\sigma_S(t,y)\right)^2\partial_x^2 + \tfrac{1}{2}\left(\beta\left(t,y\right)\right)^2\partial_y^2 \\
&\quad + \rho\alpha\left(t,y\right)x\pi(t,x,y)\sigma_S(t,y)\partial_{xy} \\
&= \left[x\left(r + \pi(t,x,y)h(t)\sigma_S(t,y)\right) - c(t,x,y)\right]\partial_x + \alpha\left(t\right)\partial_y \\
&\quad + \tfrac{1}{2}\left(x\pi(t,x,y)\sigma_S(t,y)\right)^2\partial_x^2 + \tfrac{1}{2}\left(\beta\left(t\right)\right)^2\partial_y^2 \\
&\quad + \rho\beta\left(t\right)x\pi(t,x,y)\sigma_S(t,y)\partial_{xy}\ .
\end{aligned}
$$

Also, with exponential utility we get the aggregator

$$
\begin{aligned}
F\left(c,y,U\right) &= \delta\left[f'\circ f^{-1}\right](U)\cdot\left(\left[\varphi\circ u_1\right](y,c) - f^{-1}(U)\right) \\
&= \delta\theta\left(U^{\frac{1}{\theta}}\right)^{\theta-1}\left(\exp\left(\tfrac{\gamma}{\theta}\kappa y + \tfrac{\gamma}{\theta}c\right) - U^{\frac{1}{\theta}}\right) \\
&= \delta\theta U\left(\frac{\exp\left(\tfrac{\gamma}{\theta}\kappa y + \tfrac{\gamma}{\theta}c\right)}{U^{\frac{1}{\theta}}} - 1\right)\ .
\end{aligned}
$$

2. Verify that (c^*,π^*) are equilibrium controls and U solves the Bellman equation. From Theorem 4.6, we have

$$
(26)\qquad l_i(t,s,x,y) = \eta_i(t,s)\exp\left(\gamma\kappa y + r\epsilon(t)x\right)\ ,\quad i = 1,2\ ,
$$

so the remainder terms vanish, i.e. $R_x = R_y = R_{xy} = 0$. We also have

$$U(t, x, y) = \exp\left(\gamma\kappa y + \epsilon(t)\gamma x\right) g(t) .$$

Plugging this and $R_x = R_y = R_{xy} = 0$ into the Bellman equation that U must solve, i.e. Equation (9), and dividing by $\exp\left(\gamma\kappa y + \epsilon(t)\gamma x\right)$, we obtain

$$(27) \quad g'(t) + \epsilon'(t)\gamma x g(t) = \inf_{c,\pi} \left\{ \begin{bmatrix} -\delta\theta g(t)\left(\frac{\exp\left(\frac{\gamma}{\theta}c - \epsilon(t)\frac{\gamma}{\theta}x\right)}{g(t)^{\frac{1}{\theta}}} - 1\right) \\ -\left(x\left(r + \pi h(t)\sigma_S(t,y)\right) - c\right)\epsilon(t)\gamma g(t) \\ -\frac{1}{2}\left(x\pi\sigma_S(t,y)\right)^2\epsilon(t)^2\gamma^2 g(t) \\ -\alpha(t)\gamma\kappa g(t) - \frac{1}{2}\left(\beta(t)\right)^2\gamma^2\kappa^2 g(t) \\ -\rho\beta(t)\pi x\sigma_S(t,y)\gamma^2\kappa\epsilon(t)g(t) \end{bmatrix} \right\},$$

$$g(T) = (\omega\delta)^\theta ,$$

The first order conditions for c, π read

$$0 = -\delta\gamma g(t)\frac{\exp\left(\frac{\gamma}{\theta}c - \epsilon(t)\frac{\gamma}{\theta}x\right)}{g(t)^{\frac{1}{\theta}}} + \epsilon(t)\gamma g(t) ,$$

$$0 = -xh(t)\sigma_S(t,y)\epsilon(t)\gamma g(t) - \pi\left(x\sigma_S(t,y)\right)^2\epsilon(t)^2\gamma^2 g(t)$$
$$- \rho\beta(t)x\sigma_S(t,y)\gamma^2\kappa\epsilon(t)g(t) .$$

These are satisfied by the controls $c^*(t, x, y)$ and $\pi^*(t, x, y)$ from Theorem 4.6. Plugging the controls into (27), we obtain

$$g'(t) = \tau(t)g(t) + \log\left(g(t)\right)\epsilon(t)g(t) ,$$
$$g(T) = (\delta\omega)^\theta ,$$

where

$$\tau(t) = \theta\left(\delta - \epsilon(t)\right) + \theta\epsilon(t)\log\left(\frac{\epsilon(t)}{\delta}\right) + \frac{1}{2}h(t)^2 + \frac{1}{2}\rho^2\beta(t)^2\gamma^2\kappa^2$$
$$- \alpha(t)\gamma\kappa - \frac{1}{2}\left(\beta(t)\right)^2\gamma^2\kappa^2 + \rho\beta(t)\gamma\kappa h(t) .$$

This ODE is obviously solved by the function g from Theorem 4.6. Hence, the function U from Theorem 4.6 solves the necessary Bellman equation.

   3. Verify that l_1, l_2 solve diffusion equations. Plugging (26) and the controls (c^*, π^*) into the diffusion equations that l_1 and l_2 must solve, i.e. (11)–(12), inserting $\epsilon'(t) = \epsilon(t)\left(\epsilon(t) - r\right)$, and dividing by $\exp\left(\gamma\kappa y + \epsilon(t)\gamma x\right)$, we obtain

$$\partial_t\eta_1(t, s) = \psi(t)\eta_1(t, s) ,$$
$$\eta_1(s, s) = \left(\frac{\epsilon(t)}{\delta}\right)^\theta g(s) ,$$

and

$$\partial_t\eta_2(t, T) = \psi(t)\eta_2(t, T) ,$$
$$\eta_2(T, T) = 1 ,$$

where

$$\psi(t) = \tau(t) + \epsilon(t)\log\left(g(t)\right) - \theta\left(\delta - \epsilon(t)\right) .$$

These ODE's are obviously solved by the function η_1, η_2 from Theorem 4.6. Hence, l_1, l_2 from Theorem 4.6 solve the necessary diffusion equations. $\qquad\square$

Proof of Lemma 4.7. Again we apply Itô's Lemma. We have

$$c^*(t,x) = \epsilon(t)x + \frac{\theta}{\gamma}\log\left(\frac{\epsilon(t)}{\delta}\right) + \frac{1}{\gamma}\log\left(g\left(t\right)\right) ,$$

where

$$g'(t) = \tau(t)g(t) + \log\left(g\left(t\right)\right)\epsilon(t)g(t) ,$$
$$\epsilon'(t) = \epsilon(t)(\epsilon(t) - r) .$$

We get the partial derivatives

$$\partial_t c^*(t,x) = \epsilon'(t)x + \epsilon'(t)\frac{\theta}{\gamma}\frac{1}{\epsilon(t)} + g'(t)\frac{1}{\gamma}\frac{1}{g(t)}$$

$$= \epsilon(t)(\epsilon(t) - r)x + \frac{\theta}{\gamma}(\epsilon(t) - r) + \frac{1}{\gamma}\tau(t)$$

$$\quad + \epsilon(t)\left(c^*(t,x) - \epsilon(t)x - \frac{\theta}{\gamma}\log\left(\frac{\epsilon(t)}{\delta}\right)\right)$$

$$= -r\epsilon(t)x + \frac{\theta}{\gamma}(\epsilon(t) - r) + \frac{1}{\gamma}\tau(t) - \frac{\theta}{\gamma}\epsilon(t)\log\left(\frac{\epsilon(t)}{\delta}\right)$$

$$\quad + \epsilon(t)c^*(t,x) ,$$

$$\partial_x c^*(t,x) = \epsilon(t) ,$$

$$\partial_x^2 c^*(t,x) = 0 .$$

The drift and diffusion coefficient under the equilibrium controls are given as

$$\mu^*(t,x) = rx - \frac{h(t) + \rho\beta\left(t\right)\gamma\kappa}{\epsilon(t)\gamma}h(t) - c^*(t,x) ,$$

$$\sigma^*(t) = -\frac{h(t) + \rho\gamma\kappa\beta(t)}{\gamma\epsilon(t)} .$$

Applying Itô's Lemma, we obtain

$$dc^*\left(t, X_t^{c^*,\pi^*}\right) = \left(\partial_t c^*\left(t, X_t^{c^*,\pi^*}\right) + \mu^*\left(t, X_t^{c^*,\pi^*}\right)\partial_x c^*\left(t, X_t^{c^*,\pi^*}\right)\right)dt$$

$$\quad + \sigma^*\left(t, X_t^{c^*,\pi^*}\right)\partial_x c^*\left(t, X_t^{c^*,\pi^*}\right)dW_t$$

$$= \begin{bmatrix} -\epsilon(t)rX_t^{c^*,\pi^*} + \frac{\theta}{\gamma}(\epsilon(t) - r) + \frac{1}{\gamma}\tau(t) \\ -\frac{\theta}{\gamma}\epsilon(t)\log\left(\frac{\epsilon(t)}{\delta}\right) + \epsilon(t)c^*(t, X_t^{c^*,\pi^*}) \\ +\left(rX_t^{c^*,\pi^*} - c^*(t, X_t^{c^*,\pi^*})\right)\epsilon(t) \\ -\frac{h(t)+\rho\kappa\gamma\beta(t)}{\gamma\epsilon(t)}h(t)\epsilon(t) \end{bmatrix}dt$$

$$\quad - \frac{h(t) + \rho\gamma\kappa\beta(t)}{\gamma\epsilon(t)}\epsilon(t)\,dW_t,$$

from which the claim follows. $\qquad\square$

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