Constructive convex programming

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Abstract

Working within Bishop-style constructive mathematics, we show that positive-valued, uniformly continuous, convex functions defined on convex and compact subsets of $\mathbb{R}^n$ have positive infimum. This gives rise to a separation theorem for convex sets. Based on these results, we show that the fundamental theorem of asset pricing is constructively equivalent to Markov's principle. The philosophical background behind all this is a constructively valid convex version of Brouwer's fan theorem. The emerging comprehensive yet concise overall picture of assets, infima of functions, separation of convex sets, and the fan theorem indicates that mathematics in convex environments has some innate constructive nature.

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1 Introduction

When mathematics is employed to solve real world problems—for instance in a decision-making process—the derived results should be directly applicable. But there is an issue: mathematics as we normally practice it is based on the law of excluded middle (LEM), which says that either a statement is true or its negation is true. This rule allows for proving the existence of an object by merely showing that the assumption of non-existence of that object is false. In view of applicability, deriving the existence of objects with such indirect proof methods has the disadvantage that the proof does not tell us how to find the object, as it only rejects non-existence. This is where constructive mathematics, which is crudely characterized by not using LEM—or in other words which is based on intuitionistic rather than classical logic—becomes important. A constructive existence proof of an object always comes with an algorithm to compute it. Of course, proving results constructively is often more challenging than proving the same results with LEM as an admitted proof tool. For a mathematical theorem the question arises whether there is a constructive proof of that result, and if not, how far away it is from being constructive. Constructive reverse mathematics (CRM) addresses this question.

CRM as we apply it in this survey classifies theorems by logical axioms, that is fragments of LEM[1]. Given a theorem, the idea is to find such an axiom which is sufficient and necessary to prove it constructively. This answers the question how far away the theorem is from being constructive. The information obtained is also important from an applied point of view, because if a theorem is not constructive, we would like to discover a constructive version of it, typically by requiring some additional initial information. The constructive version of the fundamental theorem of asset pricing which we derive at the end of Section 4 is an example for this.

Whereas a fair amount of mathematics has been investigated from a constructive point of view, this is not yet the case for the field of financial mathematics. One reason for this is that constructive mathematics is considered a foundational or even philosophical issue and thus far away from highly applied disciplines like finance. However, there are strong arguments for carefully looking into the proofs of applied theorems, since constructive proofs are close to algorithms for constructing the desired objects.

As an initial case study we chose to investigate the fundamental theorem of asset pricing, which says that in absence of arbitrage trading strategies there exists a martingale measure. This theorem is the backbone of mathematical finance. Since it is normally proved by means of highly non-constructive separation results we first assumed that it is far away from being constructive.

[1] Another view of CRM is to take function existence axioms and induction axioms into account as well. This is more refined. In a first approach to calibrate theorems of finance, we chose the 'simple' variant.

However, it turns out to be equivalent to Markov's principle, which is the double-negation-elimination for purely existential formulas,

$$\neg\neg\exists n\, A(n) \;\rightarrow \exists n\, A(n)\,, \tag{1}$$

where $A(n)$ is quantifier-free for each $n \in \mathbb{N}$. This principle amounts to unbounded search and is considered acceptable from a computational point of view. This equivalence was shown in [4] and is the content of Section 4. The proofs therein are quite easy. However, this is possible only since the required mathematical background was outsourced to Section 3.

The equivalence between the fundamental theorem of asset pricing and Markovs principle is based on the observation that positive-valued, uniformly continuous, convex functions defined on convex and compact subsets of $\mathbb{R}^n$ have positive infimum. This as well as a crucial consequence—a separation theorem for convex sets—was published in [3] and is worked out in Section 3.

In Section 5, which is based on [5], all theses results are traced back to a novel constructively valid version of Brouwer's fan theorem. We introduce co-convexity as a property of subsets B of $\{0,1\}^*$, the set of finite binary sequences, and prove that co-convex bars are uniform. Moreover, we establish a canonical correspondence between detachable subsets B of $\{0,1\}^*$ and uniformly continuous functions f defined on the unit interval such that B is a bar if and only if the corresponding function f is positive-valued, B is a uniform bar if and only if f has positive infimum, and B is co-convex if and only if f satisfies a weak convexity condition.

2 Bishop-style constructive mathematics

Constructive mathematics in the tradition of Errett Bishop [6, 7] is characterised by not using the law of excluded middle as a proof tool. As a major consequence, properties of the real number line $\mathbb{R}$ like the *limited principle of omniscience*

LPO $\forall x, y \in \mathbb{R}\, (x < y \,\vee\, x > y \,\vee\, x = y)$,

the *lesser limited principle of omniscience*

LLPO $\forall x, y \in \mathbb{R}\, (x \leq y \,\vee\, x \geq y)$,

and *Markov's principle*(the following formulation of Markov's principle in terms of real numbers is equivalent to the formulation (1))

MP $\forall x \in \mathbb{R}\, (\neg\, (x = 0) \;\Rightarrow\; |x| > 0)$

are no longer provable propositions but rather considered additional axioms. Many properties of the reals still hold constructively.

Lemma 1. *For all real numbers x, y, z,*

1. $x = 0 \Leftrightarrow |x| = 0$
2. $x \geq y \Leftrightarrow \neg (x < y)$
3. $x = y \Leftrightarrow \neg\neg (x = y)$
4. $x < y \Rightarrow x < z \vee z < y$
5. $|x| \cdot |y| > 0 \Leftrightarrow |x| > 0 \wedge |y| > 0$
6. $|x| > 0 \Leftrightarrow x > 0 \vee x < 0$

If $\mathbb{R}$ is replaced with $\mathbb{Q}$ in the statement of LPO, then the resulting proposition can be proved constructively. Fix an inhabited subset S of $\mathbb{R}$ (*inhabited* means that there exists s with $s \in S$) and $x \in \mathbb{R}$. The real number x is a *lower bound* of S if

$$\forall s \in S \, (x \leq s)$$

and the *infimum* of S if it is a lower bound of S and

$$\forall \varepsilon > 0 \, \exists s \in S \, (s < x + \varepsilon) \, .$$

In this case we write $x = \inf S$. The notions *upper bound*, *supremum*, and $x = \sup S$ are defined analogously. We cannot assume that every inhabited set with a lower bound has an infimum. However, under some additional conditions, this is the case. See [9, Corollary 2.1.19] for a proof of the following criterion.

Lemma 2. *Let S be an inhabited set of real numbers which has a lower bound. Assume further that for all $p, q \in \mathbb{Q}$ with $p < q$ either p is a lower bound of S or else there exists $s \in S$ with $s < q$. Then S has an infimum.*

Set $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}^+ = \{1, 2, \ldots\}$, and $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$. For $X \subseteq \mathbb{R}$, a function $h : X \to \mathbb{R}$ is *weakly increasing* if

$$\forall s, t \in X \, (s < t \;\Rightarrow\; f(s) \leq f(t))$$

and *strictly increasing* if

$$\forall s, t \in X \, (s < t \;\Rightarrow\; f(s) < f(t)) \, .$$

The properties *weakly decreasing* and *strictly decreasing* are defined analogously.

Lemma 3. *For every weakly increasing function* $h : \mathbb{N} \to \{0, 1\}$ *with* $h(0) = 0$ *the set*

$$S = \left\{3^{-k} \mid h(k) = 0\right\}$$

has an infimum. If $\inf S > 0$*, there exists* k *such that*

1. $h(k) = 0$
2. $h(k + 1) = 1$
3. $\inf S = 3^{-k}$.

Moreover,

$$h(n) = 1 \quad \Leftrightarrow \quad \inf S \geq 3^{-n+1} \quad \Leftrightarrow \quad \inf S > 3^{-n}$$

for all n.

Proof. We apply Lemma 2. Note that $1 \in S$ and that 0 is a lower bound of S. Fix $p, q \in \mathbb{Q}$ with $p < q$. If $p \leq 0$, p is a lower bound of S. Now assume that $0 < p$. Then there exists k with $2^{-k} < p$. If $h(k) = 0$, there exist $s \in S$ (choose $s = 2^{-k}$) with $s < q$. If $h(k) = 1$, we can compute the minimum s_0 of S. If $p < s_0$, p is a lower bound of S; if $s_0 < q$, there exists $s \in S$ (choose $s = s_0$) with $s < q$.

If $\inf S > 0$, there exists l such that $3^{-l} < \inf S$. Therefore, $h(l) = 1$. Let k be the largest number such that $h(k) = 0$.

Assume that $h(n) = 1$. Let l be the largest natural number with $h(l) = 0$. Then $l \leq n - 1$ and thus $\inf S = 3^{-l} \geq 3^{-n+1}$.

Assume that $\inf S > 3^{-n}$. Then there exists k with (1), (2), and (3). We obtain $k < n$ and therefore $h(n) = 1$.

□

Let X be a metric space with metric d. Fix $\varepsilon > 0$ and sets $D \subseteq C \subseteq X$. D is an *ε-approximation* of C if for every $c \in C$ there exists $d \in D$ with $d(c, d) < \varepsilon$. The set C is

- *totally bounded* if for every $\varepsilon > 0$ there exist elements $x_1, \ldots, x_m$ of C such that $\{x_1, \ldots, x_m\}$ is an ε-approximation of C
- *complete* if every Cauchy sequence in C has a limit in C
- *closed in* X if every sequence in C which converges in X also converges in C
- *compact* if it is totally bounded and complete

- *located* if it is inhabited and if for every $x \in X$ the distance

$$d(x, C) = \inf \{d(x, c) \mid c \in C\}$$

exists.

If X is complete, a subset C of X is closed in X if and only if it is complete. Note further that totally bounded sets are inhabited. Totally boundedness is another crucial criterion for the existence of suprema and infima, see [9, Proposition 2.2.5].

Lemma 4. *If $C \subseteq \mathbb{R}$ is totally bounded, then $\inf C$ and $\sup C$ exist.*

Lemma 5. *Suppose that C is a located subset of X. Then the function*

$$f : X \to \mathbb{R},\ x \mapsto d(x, C)$$

is uniformly continuous.

Proof. Fix $x, y \in X$. For every $c \in C$ we have

$$d(x, c) \leq d(x, y) + d(y, c).$$

We obtain

$$d(x, c) \leq d(x, y) + d(y, C).$$

and therefore

$$d(x, C) \leq d(x, y) + d(y, C).$$

This implies

$$f(x) - f(y) \leq d(x, y).$$

□

We learn from [9, Proposition 2.2.6] that totally boundedness is preserved by uniformly continuous functions.

Lemma 6. *If $C \subseteq X$ is totally bounded and $f : C \to Y$ is uniformly continuous, where Y is also a metric space. Then*

$$\{f(x) \mid x \in C\}$$

is totally bounded.

The following combination of Lemma 4 and Lemma 6 is used frequently.

Lemma 7. *If $C \subseteq X$ is totally bounded and $f : C \to \mathbb{R}$ is uniformly continuous, then the infimum of f,*

$$\inf f = \inf \{f(x) \mid x \in C\}$$

and the supremum of f,

$$\sup f = \sup \{f(x) \mid x \in C\}$$

exist.

We refer to [7, Chapter 4, Proposition 4.4] for a proof of the following result.

Lemma 8. *A totally bounded subset C of a metric space X is located.*

A subset C of a linear space Z is *convex* if $\lambda x + (1-\lambda)y \in C$ for all $x, y \in C$ and $\lambda \in [0,1]$. The following lemma is of great importance in Section 3.

Lemma 9. *Let Y be an inhabited convex subset of a Hilbert space H and $x \in H$ such that $d = d(x, Y)$ exists. Then there exists a unique a in the closure $\overline{Y}$ of Y such that $\|a - x\| = d$. Furthermore, for all $c \in Y$ we have*

$$\langle a - x, c - a \rangle \geq 0$$

and therefore

$$\langle a - x, c - x \rangle \geq d^2.$$

Proof. Fix a sequence (c_l) in Y such that $\|c_l - x\| \to d$. Since

$$\|c_m - c_l\|^2 = \|(c_m - x) - (c_l - x)\|^2 =$$

$$2\|c_m - x\|^2 + 2\|c_l - x\|^2 - \underbrace{4\left\|\frac{c_m + c_l}{2} - x\right\|^2}_{\geq 4d^2} \leq$$

$$2\left(\|c_m - x\|^2 - d^2\right) + 2\left(\|c_l - x\|^2 - d^2\right),$$

(c_l) is a Cauchy sequence and therefore converges to an $a \in \overline{Y}$. Since $\|c_l - x\| \to \|a - x\|$, we obtain $\|a - x\| = d$. Now fix $b \in \overline{Y}$ with $\|b - x\| = d$. Then

$$\|a - b\|^2 = \|(a - x) - (b - x)\|^2 =$$

$$2\|a - x\|^2 + 2\|b - x\|^2 - \underbrace{4\left\|\frac{a + b}{2} - x\right\|^2}_{\geq 4d^2} \leq 0,$$

thus $a = b$.

Fix $c \in Y$ and $\lambda \in (0,1)$. Since

$$\|a-x\|^2 \leq \|(1-\lambda)a + \lambda c - x\|^2 = \|(a-x) + \lambda(c-a)\|^2 =$$
$$\|a-x\|^2 + \lambda^2 \|c-a\|^2 + 2\lambda\langle a-x, c-a\rangle,$$

we obtain

$$0 \leq \lambda \|c-a\|^2 + 2\langle a-x, c-a\rangle .$$

Since λ can be arbitrarily small, we can conclude that

$$\langle a-x, c-a\rangle \geq 0 .$$

This also implies that

$$\langle a-x, c-x\rangle = \langle a-x, c-a\rangle + \langle a-x, a-x\rangle \geq d^2 .$$

□

For $x, y \in \mathbb{R}^n$, we define the scalar product $\langle x, y\rangle = \sum_{i=1}^n x_i \cdot y_i$, the norm $\|x\| = \sqrt{\langle x, x\rangle}$, and the metric $d(x,y) = \|y-x\|$.

Vectors $x_1, \dots, x_n \in \mathbb{R}^n$ are *linearly independent* if for all $\lambda \in \mathbb{R}^n$ the implication

$$\sum_{i=1}^n |\lambda_i| > 0 \Rightarrow \|\sum_{i=1}^n \lambda_i x_i\| > 0$$

is valid. Such vectors span located subsets [9, Lemma 4.1.2].

Lemma 10. *If $x_1, \dots, x_m \in \mathbb{R}^n$ are linearly independent, then the set*

$$\left\{ \sum_{i=1}^m \xi_i x_i \mid \xi \in \mathbb{R}^m \right\}$$

is closed in $\mathbb{R}^n$ and located.

3 Convexity and constructive infima

Fix $n \in \mathbb{N}^+$. In this section, the variable i denotes elements of $\{1, \dots, n\}$. A function $f : C \to \mathbb{R}$, where C is a convex subset of $\mathbb{R}^n$, is called *quasi-convex* if

$$f(\lambda x + (1-\lambda)y) \leq \max(f(x), f(y))$$

for all $\lambda \in [0,1]$ and $x, y \in C$. Hence, in particular, any convex function $f : C \to \mathbb{R}$ — a function is *convex* if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $\lambda \in [0,1]$ and $x, y \in C$— is quasi-convex.

Theorem 1. *If $C \subseteq \mathbb{R}^n$ is compact and convex and*

$$f : C \to \mathbb{R}^+$$

is quasi-convex and uniformly continuous, then $\inf f > 0$.

In order to prove Theorem 1, we start with some technical lemmas. For a subset C of $\mathbb{R}^n$ and $t \in \mathbb{R}$ we define

$$C_i^t = \{x \in C \mid x_i = t\} .$$

Lemma 11. *Fix a convex subset C of $\mathbb{R}^n$ and $t \in \mathbb{R}$. Suppose further that there are $y, z \in C$ with $y_i < t < z_i$. Then there exists $\lambda \in \,]0, 1[$ such that*

$$\lambda y + (1 - \lambda) z \in C_i^t .$$

Proof. Set $\lambda = \frac{z_i - t}{z_i - y_i}$. □

We call C_i^t *admissible* if there exist $y, z \in C$ with $y_i < t < z_i$.

Lemma 12. *Let $n > 1$. Fix a subset C of $\mathbb{R}^n$ and suppose that C_i^t is convex and compact. Then there exists a convex compact subset $\hat{C}$ of $\mathbb{R}^{n-1}$ and a uniformly continuous bijection*

$$g : \hat{C} \to C_i^t$$

which is affine *in the sense that*

$$g(\lambda x + (1 - \lambda) y) = \lambda g(x) + (1 - \lambda) g(y)$$

for all $\lambda \in [0, 1]$ and $x, y \in \hat{C}$.

Proof. We can assume that $i = 1$. Set

$$\hat{C} = \left\{ (x_2, \ldots, x_n) \in \mathbb{R}^{n-1} \mid (t, x_2, \ldots, x_n) \in C_1^t \right\}$$

and

$$g(x_2, \ldots, x_n) = (t, x_2, \ldots, x_n) .$$

□

The next lemma is crucial for the proof of Theorem 1, and of interest on its own.

Lemma 13. *If $C \subseteq \mathbb{R}^n$ is convex and compact and C_i^t is admissible, then C_i^t is convex and compact.*

Proof. Let $C \subseteq \mathbb{R}^n$ be convex and compact and let C_i^t be admissible. Without loss of generality, we may assume that $t = 0$ and $i = 1$. There exist $y, z \in \mathrm{C}$ with $y_1 < 0 < z_1$. Define

$$\mathcal{M} = C_1^0,\ \mathcal{L} = \{x \in C \mid x_1 \leq 0\},\ \mathcal{R} = \{x \in C \mid x_1 \geq 0\}.$$

We show that the sets $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{M}$ are convex and compact. It is clear that these sets are convex and complete. By applying Lemma 6 repeatedly, we show that they are totally bounded as well.
We start with the case of $\mathcal{R}$. Set

$$\kappa : \mathbb{R} \to \mathbb{R},\ s \mapsto \max(-s, 0)$$

and

$$f : \mathbb{R}^n \to \mathbb{R}^n,\ x \mapsto \frac{z_1}{z_1 + \kappa(x_1)} x + \frac{\kappa(x_1)}{z_1 + \kappa(x_1)} z$$

and note that

- f is uniformly continuous
- f maps C onto $\mathcal{R}$.

In order to prove the latter, we proceed step by step and show that

1. $f(C) \subseteq C$
2. $f(C) \subseteq \mathcal{R}$
3. $f(C) = \mathcal{R}$.

The property (1) follows from the convexity of C. In order to show (2), fix $x \in C$. We show that the assumption that the first component of $f(x)$ is negative is contradictory. So assume that

$$\frac{z_1}{z_1 + \kappa(x_1)} x_1 + \frac{\kappa(x_1)}{z_1 + \kappa(x_1)} z_1 < 0.$$

Then $x_1 < 0$ and therefore $\kappa(x_1) = -x_1$. We obtain

$$z_1 \cdot x_1 - x_1 \cdot z_1 < 0,$$

a contradiction. The property (3) follows from the fact that f leaves the elements of $\mathcal{R}$ unchanged. So we have shown that $\mathcal{R}$ is totally bounded. Analogously, we can show that $\mathcal{L}$ is totally bounded. Next, we show that

$$\mathcal{M} = f(\mathcal{L}),$$

which implies that $\mathcal{M}$ is totally bounded as well. To this end, fix $x \in \mathcal{L}$. Then $\kappa(x_1) = -x_1$ and therefore

$$\frac{z_1}{z_1 + \kappa(x_1)} x_1 + \frac{\kappa(x_1)}{z_1 + \kappa(x_1)} z_1 = 0,$$

which implies that $f(x) \in \mathcal{M}$. □

The following Lemma 14 basically already proves Theorem 1.

Lemma 14. *Fix a convex compact subset C of $\mathbb{R}^n$ and suppose that*

$$f : C \to \mathbb{R}^+$$

is quasi-convex and uniformly continuous. Assume further that

$$\inf \left\{ f(x) \mid x \in C_i^t \right\} > 0$$

for every admissible C_i^t. Then $\inf f > 0$.

Proof. Note that $\inf f$ exists by Lemma 7. We define a sequence (x^m) in C and a binary sequence (λ^m) such that

- $\lambda^{m+1} = 0 \Rightarrow \lambda^m = 0$ and $f(x^{m+1}) < \min\left(2^{-(m+1)}, f(x^m)\right)$
- $\lambda^{m+1} = 1 \Rightarrow \inf f > 0$ and $x^{m+1} = x^m$

for every m. Note that under these conditions the sequence $(f(x^m))$ is weakly decreasing.

Let x^0 be an arbitrary element of C and set $\lambda^0 = 0$. Assume that x^m and λ^m have already been defined.

<u>case 1</u> If $\lambda^m = 1$, set $x^{m+1} = x^m$ and $\lambda^{m+1} = 1$.

<u>case 2</u> If $\lambda^m = 0$ and $0 < \inf f$, set $x^{m+1} = x^m$ and $\lambda^{m+1} = 1$.

<u>case 3</u> If $\lambda^m = 0$ and

$$\inf f < \min\left(2^{-(m+1)}, f(x^m)\right),$$

choose x^{m+1} in C with

$$f(x^{m+1}) < \min\left(2^{-(m+1)}, f(x^m)\right)$$

and set $\lambda^{m+1} = 0$.

We show that the sequence (x^m) converges.

It is sufficient to show that for each component i the sequence $(x_i^m)_{m\in\mathbb{N}}$ is a Cauchy sequence. We consider the case $i = 1$. Fix $\varepsilon > 0$. Let D be the image of C under the projection onto the first component, i.e.

$$D = \{x_1 \mid x \in C\}\,.$$

Note that D is a totally bounded interval. Denote its infimum by a and its supremum by b.

<u>case 1</u> If $b - a < \varepsilon$, then $\left|x_1^k - x_1^l\right| \leq \varepsilon$ for all k, l.

<u>case 2</u> If $b - a > 0$, there exists a finite $\frac{\varepsilon}{2}$-approximation F of $]a, b[$. Note that for every t with $a < t < b$ the set $\mathcal{C}_1^t$ is admissible. Hence, we can choose an l_0 such that

$$f(x) > 2^{-l_0}$$

for all $t \in F$ and all $x \in \mathcal{C}_1^t$. Fix $k, l \geq l_0$. We show that $\left|x_1^k - x_1^l\right| \leq \varepsilon$.

<u>case 2.1</u> If $\lambda^{l_0} = 1$, then $x^k = x^l$.

<u>case 2.2</u> If $\lambda^{l_0} = 0$, then $f(x^k) < 2^{-l_0}$ and $f(x^l) < 2^{-l_0}$. Suppose that $x_1^k - x_1^l > \varepsilon$. Then there exists $t \in F$ with $x_1^k < t < x_1^l$. According to Lemma 11 there is $\mu \in]0, 1[$ such that $\mu x^k + (1-\mu)x^l \in C_1^t$, and by quasi-convexity of f we obtain

$$f(\mu x^k + (1-\mu)x^l) \leq \max\left(f(x^k), f(x^l)\right) < 2^{-l_0}$$

which contradicts the construction of l_0. Therefore, $x_1^k - x_1^l \leq \varepsilon$, and similarly also $x_1^l - x_1^k \leq \varepsilon$.

Let $x \in C$ be the limit of the sequence (x^m). There exists l such that

$$f(x) > 2^{-l}$$

and a k such that

$$d(x, y) < 2^{-k} \;\Rightarrow\; |f(x) - f(y)| < 2^{-(l+1)}$$

for all $y \in C$. Finally, pick $N > l$ such that

$$d(x, x^N) < 2^{-k}.$$

Then $f(x^N) \geq 2^{-N}$, therefore $\lambda_N = 1$, which implies that $\inf f > 0$. □

Proof of Theorem 1. We use induction over the dimension n.

If $n = 1$, then every admissible set C_1^t equals $\{t\}$, so $\inf f > 0$ follows from Lemma 14.

Now fix $n > 1$ and assume the assertion of Theorem 1 holds for $n-1$. Furthermore, let C be a convex compact subset of $\mathbb{R}^n$, and suppose that

$$f : C \to \mathbb{R}^+$$

is convex and uniformly continuous. Fix an admissible subset C_i^t of C. By Lemma 13, C_i^t is convex and compact. Using Lemma 12 construct the convex compact set $\hat{C} \subseteq \mathbb{R}^{n-1}$ and the uniformly continuous affine bijection

$$g : \hat{C} \to C_i^t.$$

Then $F : \hat{C} \to \mathbb{R}^+$ given by $F = f \circ g$ is quasi-convex and uniformly continuous. The induction hypothesis now implies that

$$\inf \left\{ f(x) \mid x \in C_t^i \right\} = \inf \left\{ F(x) \mid x \in \hat{C} \right\} > 0\,.$$

Thus, $\inf f > 0$ follows from Lemma 14. □

Theorem 2. *Let C and Y be subsets of $\mathbb{R}^n$ and suppose that*

1. *C is convex and compact*
2. *Y is convex, complete, and located*
3. *$d(c, y) > 0$ for all $c \in C$ and $y \in Y$.*

Then there exist $p \in \mathbb{R}^n$ and reals α, β such that

$$\langle p, c\rangle < \alpha < \beta < \langle p, y\rangle$$

for all $c \in C$ and $y \in Y$. In particular, the sets C and Y are strictly separated by the hyperplane

$$H = \{x \in \mathbb{R}^n \mid \langle p, x\rangle = \gamma\}\,,$$

with $\gamma = \frac{1}{2}(\alpha + \beta)$.

Proof. By Lemma 5, the function

$$f : C \to \mathbb{R},\ c \mapsto d(c, Y)$$

is uniformly continuous. Since Y is closed, Lemma 9 implies that for every $c \in C$ there is a unique $y \in Y$ with

$$f(c) = d(c, y).$$

Therefore, f is positive-valued and also convex, as we can see as follows. Fix $c_1, c_2 \in C$ and $\lambda \in [0,1]$. There are $y_0, y_1, y_2 \in Y$ such that

$$f(c_1) = d(c_1, y_1)\,, \quad f(c_2) = d(c_2, y_2)\,,$$

and

$$f(\lambda c_1 + (1-\lambda)c_2) = d(\lambda c_1 + (1-\lambda)c_2, y_0)\,.$$

We obtain

$$\begin{aligned} f(\lambda c_1 + (1-\lambda)c_2) &= d(\lambda c_1 + (1-\lambda)c_2, y_0) \\ &\leq d(\lambda c_1 + (1-\lambda)c_2, \lambda y_1 + (1-\lambda)y_2) \\ &\leq \lambda d(c_1, y_1) + (1-\lambda)d(c_2, y_2) \\ &= \lambda f(c_1) + (1-\lambda)f(c_2)\,. \end{aligned}$$

By Theorem 1, $\inf f > 0$. The set

$$Z = \{y - c \mid x \in C,\, y \in Y\}$$

is inhabited and convex. Since we have

$$\inf\{\, \|y - c\| \ \mid x \in C,\, y \in Y\} = \inf f\,,$$

we can conclude that $\delta = d(0, Z)$ exists and is positive. By Lemma 9, there exists $p \in \mathbb{R}^n$ such that

$$\langle p, y\rangle \geq \delta^2 + \langle p, c\rangle$$

for all $y \in Y$ and $c \in C$. By Lemma 7, $\eta = \sup\{\langle p, c\rangle \mid c \in C\}$ exists. Setting

$$\alpha = \frac{\delta^2}{3} + \eta \ \text{ and } \ \beta = \frac{\delta^2}{2} + \eta\,,$$

we obtain

$$\langle p, c\rangle < \alpha < \beta < \langle p, y\rangle$$

for all $c \in C$ and $y \in Y$. □

4 The fundamental theorem of asset pricing

Fix $m, n \in \mathbb{N}^+$. Set $I_n = \{1, \ldots, n\}$. In this section, the variable i always stands for an element of I_m and the variable j always stands for an element of I_n. The linear space of real matrices A with m rows and n columns is denoted by $\mathbb{R}^{m\times n}$. For such an A, we denote its entry at row i and column j by a_{ij}. For $l \in \mathbb{N}^+$ and $B \in \mathbb{R}^{n\times l}$, we denote the matrix product of A and

B (which is an element of $\mathbb{R}^{m\times l}$) by $A \cdot B$. Let C be the convex hull of the unit vectors of $\mathbb{R}^n$.

Our market consists of m assets. Their value at time 0 (present) is known. Their value at time 1 (future) is unknown. There are n possible developments and we know the prices in each of the n cases. Define a matrix $A \in \mathbb{R}^{m\times n}$ as follows: the value of the entry a_{ij} is the price development (price at time 1 minus price at time 0) of asset i in case j. Set

$$P = \left\{ p \in \mathbb{R}^n \mid \sum_{j=1}^m p_j = 1 \text{ and } 0 < p_j \text{ for all } j \right\}.$$

A vector $p \in P$ is a *martingale measure* if $A \cdot p = 0$. Under a martingale measure the average profit is zero, that is todays price of the assets is reasonable in the sense of being the expected value of the assets tomorrow. For $x \in \mathbb{R}^n$ we define

$$x > 0 \quad :\Leftrightarrow \quad \forall j \, (x_j \geq 0) \ \wedge \ \exists j \, (x_j > 0) \, .$$

A vector $\xi \in \mathbb{R}^m$ is an *arbitrage trading strategy* if $\xi \cdot A > 0$.
Note that every $\xi \in \mathbb{R}^m$ corresponds to a trading strategy, where ξ_i denotes the number of shares of asset i that the trader buys. Hence, the payoff at time 1 over all possible future scenarios is $\xi \cdot A$. Thus arbitrage strategies are trading strategies which correspond to riskless gains, since they never produce any losses, and even a strict gain for at least one possible future scenario.
The *fundamental theorem of asset pricing* says that the absence of an arbitrage trading strategy is equivalent to the existence of a martingale measure.

FTAP Fix an $\mathbb{R}^{m\times n}$-matrix A with linearly independent rows. Then

$$\neg \exists \xi \in \mathbb{R}^m \ (\xi \cdot A > 0) \ \Leftrightarrow \ \exists p \in P \ (A \cdot p = 0) \, .$$

Note that "$\Leftarrow$" is clear: assume there exist both p in P with $A \cdot p = 0$ and ξ in $\mathbb{R}^m$ with $\xi \cdot A > 0$. Then from $A \cdot p = 0$ we can conclude that $\xi \cdot A \cdot p = 0$ and from $\xi \cdot A > 0$ and $p \in P$ we can conclude that $\xi \cdot A \cdot p > 0$. This is a contradiction.

Theorem 3.

$$\text{FTAP} \Leftrightarrow \text{MP}$$

Proof. Fix an $\mathbb{R}^{m\times n}$-matrix A such that

$$\neg \exists \xi \in \mathbb{R}^m \ (\xi \cdot A > 0) \, .$$

Let Y be the linear subspace of $\mathbb{R}^n$ which is generated by the rows of A. This set is convex. Since we have assumed that the rows of A are linearly independent, we can conclude from Lemma 10 that Y is convex, closed, and located. The set C is convex and compact. By MP, we obtain

$$\forall c \in C, y \in Y \,(d(c, y) > 0) \,.$$

By Theorem 2, there exists a vector $p \in \mathbb{R}^n$ and reals α, β such that

$$\forall y \in Y, c \in C \,(\langle p, c\rangle > \alpha > \beta > \langle p, y\rangle) \,.$$

This implies that $A \cdot p = 0$ and that all components of p are positive. We can assume further that $p_1 + \ldots + p_n = 1$.

In order to prove the converse direction, fix a real number a with $\neg(a = 0)$. Apply FTAP to the matrix

$$A = (|a|\,, -1).$$

Note that A has linearly independent rows. The no-arbitrage condition is satisfied: assume that there exists $\xi \in \mathbb{R}$ with

$$(\xi \cdot |a|\,, -\xi) > 0. \tag{2}$$

We obtain that $\xi \cdot |a| \geq 0$ and $\xi \leq 0$, which implies that $\xi = 0$, a contradiction to (2). Now FTAP yields the existence of a $p \in P$ with

$$p_1 \cdot |a| = p_2.$$

This implies that $|a| > 0$.

□

Note that we obtain the following constructively valid version of FTAP, see [4, Corollary 3].

Theorem 4. *Fix an $\mathbb{R}^{m \times n}$-matrix A with linearly independent rows. Then*

$$\forall c \in C,\, \xi \in \mathbb{R}^m \,(d(c, \xi \cdot A) > 0 \;\Rightarrow\; \exists p \in P \;(A \cdot p = 0)) \,.$$

This theorem says that we can construct a martingale measure if we exclude the existence of arbitrage strategies in a stricter way.

5 Brouwer's fan theorem and convexity

We write $\{0,1\}^*$ for the set of all finite binary sequences u, v, w. Let $\emptyset$ be the empty sequence and let $\{0,1\}^{\mathbb{N}}$ be the set of all infinite binary sequences α, β, γ. For every u let $|u|$ be the *length* of u, that is $|\emptyset| = 0$ and for $u = (u_0, \ldots, u_{n-1})$ we have $|u| = n$. For $v = (v_0, \ldots, v_{m-1})$, the *concatenation* $u * v$ of u and v is defined by

$$u * v = (u_0, \ldots, u_{n-1}, v_0, \ldots, v_{m-1}).$$

The *restriction* $\overline{\alpha}n$ of α to n bits is given by

$$\overline{\alpha}n = (\alpha_0, \ldots, \alpha_{n-1}).$$

Thus $|\overline{\alpha}n| = n$ and $\overline{\alpha}0 = \emptyset$. For u with $n \leq |u|$, the restriction $\overline{u}n$ is defined analogously. A subset B of $\{0,1\}^*$ is *closed under extension* if $u*v \in B$ for all $u \in B$ and for all v. A sequence α *hits* B if there exists n such that $\overline{\alpha}n \in B$. B is a *bar* if every α hits B. B is a *uniform bar* if there exists N such that for every α there exists $n \leq N$ such that $\overline{\alpha}n \in B$. Often one requires B to be *detachable*, that is for every u the statement $u \in B$ is decidable. Now we are ready to introduce Brouwer's *fan theorem for detachable bars.*

FAN Every detachable bar is a uniform bar.

In Bishop's constructive mathematics, FAN is neither provable nor falsifiable, see [8, Section 3 of Chapter 5]. In their seminal paper [11], Julian and Richman established a correspondence between FAN and functions on $[0,1]$ as follows.

Proposition 1. *For every detachable subset B of $\{0,1\}^*$ there exists a uniformly continuous function $f : [0,1] \to [0,\infty[$ such that*

1. *B is a bar $\Leftrightarrow$ f is positive-valued*
2. *B is a uniform bar $\Leftrightarrow$ f has positive infimum.*

Conversely, for every uniformly continuous function $f : [0,1] \to [0,\infty[$ there exists a detachable subset B of $\{0,1\}^*$ such that (1) and (2) hold.

Consequently, FAN is equivalent to the statement that every uniformly continuous, positive-valued function defined on the unit interval has positive infimum. Now, in view of Theorem 1, the question arises whether there is a constructively valid 'convex' version of the fan theorem. To this end, we define

$$u < v \;:\Leftrightarrow\; |u| = |v| \wedge \exists k < |u| \, (\overline{u}k = \overline{v}k \wedge u_k = 0 \wedge v_k = 1)$$

and

$$u \leq v :\Leftrightarrow u = v \vee u < v.$$

A subset B of $\{0,1\}^*$ is *co-convex* if for every α which hits B there exists n such that either

$$\{v \mid v \leq \overline{\alpha}n\} \subseteq B \quad \text{or} \quad \{v \mid \overline{\alpha}n \leq v\} \subseteq B\,.$$

Note that, for detachable B, co-convexity follows from the convexity of the complement of B, where $C \subseteq \{0,1\}^*$ is *convex* if for all u, v, w we have

$$u \leq v \leq w \wedge u, w \in C \Rightarrow v \in C.$$

Define the *upper closure* B' of B by

$$B' = \{u \mid \exists k \leq |u| \, (\overline{u}k \in B)\}\,.$$

Note that B is a (detachable) bar if and only if B' is a (detachable) bar and B is a uniform bar if and only B' is a uniform bar. Therefore, we may assume that bars are closed under extension.

Theorem 5. *Every co-convex bar is a uniform bar.*

Proof. Fix a co-convex bar B. Since the upper closure of B is also co-convex, we can assume that B is closed under extension. Define

$$C = \{u \mid \exists n \, \forall w \in \{0,1\}^n \, (u * w \in B)\}\,.$$

Note that $B \subseteq C$ and that C is closed under extension as well. Moreover, B is a uniform bar if and only if there exists n such that $\{0,1\}^n \subseteq C$.
First, we show that

$$\forall u \, \exists i \in \{0,1\} \, (u * i \in C)\,. \tag{3}$$

Fix u. For

$$\beta = u * 1 * 0 * 0 * 0 * \ldots$$

there exist an l such that either

$$\left\{v \mid v \leq \overline{\beta}l\right\} \subseteq B,$$

or

$$\left\{v \mid \overline{\beta}l \leq v\right\} \subseteq B.$$

Since B is closed under extension, we can assume that $l > |u| + 1$. Let $m = l - |u| - 1$. If $\left\{v \mid v \leq \overline{\beta}l\right\} \subseteq B$, we can conclude that

$$u * 0 * w \in B$$

for every w of length m, which implies that $u * 0 \in C$. If $\{v \mid \overline{\beta}l \leq v\} \subseteq B$, we obtain

$$u * 1 * w \in B$$

for every w of length m, which implies that $u * 1 \in C$. This concludes the proof of (3).

By countable choice, there exists a function $F : \{0,1\}^* \to \{0,1\}$ such that

$$\forall u \; (u * F(u) \in C) \, .$$

Define α by

$$\alpha_n = 1 - F(\overline{\alpha}n).$$

Next, we show by induction on n that

$$\forall n \, \forall u \in \{0,1\}^n \, (u \neq \overline{\alpha}n \Rightarrow u \in C) \, . \tag{4}$$

If $n = 0$, the statement clearly holds, since in this case the statement $u \neq \overline{\alpha}n$ is false. Now fix some n such that (4) holds. Moreover, fix $w \in \{0,1\}^{n+1}$ such that $w \neq \overline{\alpha}(n+1)$.

case 1. $\overline{w}n \neq \overline{\alpha}n$. Then $\overline{w}n \in C$ and therefore $w \in C$.

case 2. $w = \overline{\alpha}n * (1 - \alpha_n) = \overline{\alpha}n * F(\overline{\alpha}n)$. This implies $w \in C$. So we have established (4).

There exists n such that $\overline{\alpha}n \in B$. Applying (4) to this n, we can conclude that every u of length n is an element of C, thus B is a uniform bar. □

Remark. *Note that we do not need to require that the co-convex bar in Theorem 5 is detachable.*

In order to include convexity in the list of Proposition 1, we introduce a notion of weakly convex functions. Let S be a subset of $\mathbb{R}$. A function $f : S \to \mathbb{R}$ is *weakly convex* if for all $t \in S$ with $f(t) > 0$ there exists $\varepsilon > 0$ such that either

$$\forall s \in S \, (s \leq t \quad \Rightarrow \quad f(s) \geq \varepsilon)$$

or

$$\forall s \in S \, (t \leq s \quad \Rightarrow \quad f(s) \geq \varepsilon) \, .$$

Remark. *Fix a dense subset D of $[0,1]$. A uniformly continuous function $f : [0,1] \to \mathbb{R}$ is weakly convex if and only its restriction to D is weakly convex.*

The following generalisation of Proposition 1 links Theorem 1 with Theorem 5.

Theorem 6. *For every detachable subset B of $\{0,1\}^*$ which is closed under extension there exists a uniformly continuous function $f : [0,1] \to \mathbb{R}$ such that*

1. *B is a bar $\Leftrightarrow$ f is positive-valued*
2. *B is a uniform bar $\Leftrightarrow$ $\inf f > 0$*
3. *B is co-convex $\Leftrightarrow$ f is weakly convex.*

Conversely, for every uniformly continuous function $f : [0,1] \to \mathbb{R}$ there exists a detachable subset B of $\{0,1\}^*$ which is closed under extension such that (1), (2), and (3) hold.

We split the proof of Theorem 6 into two parts.

Part I: Construction of a function f for given B

Fix a detachable subset B of $\{0,1\}^*$ which is closed under extension. We can assume that $\emptyset \notin B$. (Otherwise, let f be the constant function $t \mapsto 1$.) First, we define a function $g : [0,1] \to \mathbb{R}$ which satisfies the properties (1) and (2) of Theorem 6. Then, we introduce a refined version f of g which satisfies all properties of Theorem 6. Define metrics

$$d_1(s,t) = |s-t|, \quad d_2((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

on $\mathbb{R}$ and $\mathbb{R}^2$, respectively. The mapping

$$(\alpha, \beta) \mapsto \inf \left\{2^{-k} \mid \overline{\alpha}k = \overline{\beta}k\right\}$$

is a compact metric on $\{0,1\}^{\mathbb{N}}$. See [8, Section 1 of Chapter 5] for an introduction to basic properties of this metric space. Define a uniformly continuous function $\kappa : \{0,1\}^{\mathbb{N}} \to [0,1]$ by

$$\kappa(\alpha) = 2 \cdot \sum_{k=0}^{\infty} \alpha_k \cdot 3^{-(k+1)}.$$

The next lemma immediately follows from the definition of κ.

Lemma 15. *For all α, β and n, we have*

- $\overline{\alpha}n = \overline{\beta}n \;\Rightarrow\; |\kappa(\alpha) - \kappa(\beta)| \leq 3^{-n}$
- $\overline{\alpha}n = \overline{\beta}n \,\wedge\, \alpha_n < \beta_n \;\Rightarrow\; \kappa(\alpha) + 3^{-(n+1)} \leq \kappa(\beta)$

- $\overline{\alpha}n \neq \overline{\beta}n \;\Rightarrow\; |\kappa(\alpha) - \kappa(\beta)| \geq 3^{-n}$
- $\overline{\alpha}n < \overline{\beta}n \;\Rightarrow\; \kappa(\alpha) < \kappa(\beta)$.

Now define

$$\eta_B : \{0,1\}^{\mathbb{N}} \to [0,1]\,,\ \alpha \mapsto \inf\left\{3^{-k} \mid \overline{\alpha}k \notin B\right\}.$$

The following lemma is an immediate consequence of Lemma 3.

Lemma 16. *The function η_B is well-defined—the infimum in the definition of η_B always exists—and uniformly continuous. If $\eta_B(\alpha) > 0$, there exists k such that*

1. $\overline{\alpha}k \notin B$
2. $\overline{\alpha}(k+1) \in B$
3. $\eta_B(\alpha) = 3^{-k}$.

Moreover,

$$\overline{\alpha}n \in B \;\Leftrightarrow\; \eta_B(\alpha) \geq 3^{-n+1} \;\Leftrightarrow\; \eta_B(\alpha) > 3^{-n}$$

for all α and n.

Set

$$C = \left\{\kappa(\alpha) \mid \alpha \in \{0,1\}^{\mathbb{N}}\right\}$$

and

$$K = \left\{(\kappa(\alpha), \eta_B(\alpha)) \mid \alpha \in \{0,1\}^{\mathbb{N}}\right\}.$$

Lemma 17. *The sets C and K are compact.*

Proof. Both sets are uniformly continuous images of the compact set $\{0,1\}^{\mathbb{N}}$ and therefore totally bounded, by Lemma 6. Suppose that $\kappa(\alpha^n)$ converges to t and $\eta_B(\alpha^n)$ converges to s. By Lemma 15, the sequence (α^n) is Cauchy, therefore it converges to a limit α. Then $\kappa(\alpha^n)$ converges to $\kappa(\alpha)$ and $\eta_B(\alpha^n)$ converges to $\eta_B(\alpha)$. Therefore $t = \kappa(\alpha)$ and $s = \eta_B(\alpha)$. Thus we have shown that both C and K are complete. □

In the following, we will use Bishop's lemma, see [7, Ch. 4, Lemma 3.8].

Lemma 18. *Let A be a complete, located subset of a metric space X, and x a point of X. Then there exists a point a in A such that $d(x,a) > 0$ entails $d(x,A) > 0$.*

Define

$$g : [0,1] \to [0,\infty[\,,\ t \mapsto d_2((t,0),K).$$

Proposition 2. *1. B is a bar* $\Leftrightarrow$ *g is positive-valued*

2. B is a uniform bar $\Leftrightarrow$ $\inf g > 0$

Proof. Assume that B is a bar. Fix $t \in [0,1]$. In view of Bishop's lemma and the compactness of K, it is sufficient to show that

$$d_2((t,0),(\kappa(\alpha),\eta_B(\alpha))) > 0$$

for each α. This follows from $\eta_B(\alpha) > 0$.

Now assume that g is positive-valued. Fix α. Since

$$d_2((\kappa(\alpha),0),K) = g(\kappa(\alpha)) > 0,$$

we can conclude that

$$d_2((\kappa(\alpha),0),(\kappa(\alpha),\eta_B(\alpha))) > 0.$$

Thus $\eta_B(\alpha)$ is positive which implies that α hits B.

The second equivalence follows from Lemma 16 and the fact that $\inf g = \inf \eta_B$. □

Set

$$-C = \{t \in [0,1] \mid d_1(t,C) > 0\}\,.$$

We would like to include the statement

- B is co-convex $\Leftrightarrow$ g is weakly convex

into Proposition 2. Note, however, that g is positive on $-C$. Thus we introduce a new function f by

$$f : [0,1] \to \mathbb{R},\ t \mapsto g(t) - d_1(t,C).$$

The next lemma lists up a few properties of f and g.

Lemma 19. *For all* α*,* n*, and* t *we have*

- $g(\kappa(\alpha)) = f(\kappa(\alpha)) \leq \eta_B(\alpha)$
- $f(\kappa(\alpha)) > 3^{-n} \;\Rightarrow\; \overline{\alpha}n \in B$
- $\overline{\alpha}n \in B \;\Rightarrow\; f(\kappa(\alpha)) \geq 3^{-n}$

- $d_1(t, C) \leq g(t)$.

Next, we clarify how f behaves on $-C$.

Lemma 20. *The set $-C$ is dense in $[0,1]$. For every $t \in -C$ there exist unique elements a, a' of C such that*

1. $t \in \,]a, a'[\, \subseteq -C$.
2. $d_1(t, C) = \min\left(d_1(t, a), d_1(t, a')\right)$.

Moreover, setting $\gamma = \kappa^{-1}(a)$ and $\gamma' = \kappa^{-1}(a')$, we obtain

3. $\forall n \left(\overline{\gamma} n \in B \ \wedge \ \overline{\gamma'} n \in B \ \Rightarrow \ f(t) \geq 3^{-n}\right)$
4. *if $d_1(t, a) < d_1(t, a')$, then*
$$\gamma \textit{ hits } B \ \Leftrightarrow \ f(t) > 0 \ \Leftrightarrow \ \inf\{f(s) \mid a \leq s \leq t\} > 0$$
5. *if $d_1(t, a') < d_1(t, a)$, then*
$$\gamma' \textit{ hits } B \ \Leftrightarrow \ f(t) > 0 \ \Leftrightarrow \ \inf\{f(s) \mid t \leq s \leq a'\} > 0.$$

Proof. Fix $t \in [0,1]$ and $\delta > 0$. If $d_1(t, C) > 0$, then $t \in -C$. Now assume that there exists α such that $d_1(t, \kappa(\alpha)) < \delta/2$. There exists u such that $d_1(\kappa(\alpha), t_u) < \delta/2$ where

$$t_u = \tfrac{1}{2} \cdot \kappa(u * 0 * 1 * 1 * 1 * \ldots) + \tfrac{1}{2} \cdot \kappa(u * 1 * 0 * 0 * 0 * \ldots).$$

Note that $t_u \in -C$ and that $d_1(t, t_u) < \delta$. So $-C$ is dense in $[0,1]$.

Fix $t \in -C$. Since for any α it is decidable whether $\kappa(\alpha) > t$ or $\kappa(\alpha) < t$, the sets $C_{<t} = \{s \in C \mid s < t\}$ and $C_{>t} = \{s \in C \mid s > t\}$ are compact. Let a be the maximum of $C_{<t}$ and let a' be the minimum of $C_{>t}$. Clearly, a and a' fulfil (1) and (2).

In order to show (3), assume that $\overline{\gamma} n \in B$ and $\overline{\gamma'} n \in B$. Fix α. We show that

$$d_2((t, 0), (\kappa(\alpha), \eta_B(\alpha))) - d_1(t, C) \geq 3^{-n}. \tag{5}$$

First, assume that $\kappa(\alpha) < t$. Then we have

$$d_2((t, 0), (\kappa(\alpha), \eta_B(\alpha))) - d_1(t, C) \geq \kappa(\gamma) - \kappa(\alpha) + \eta_B(\alpha).$$

If $\overline{\alpha} n = \overline{\gamma} n$, then $\overline{\alpha} n \in B$ and we can conclude that $\eta_B(\alpha) \geq 3^{-n+1}$, by Lemma 16. On the other hand, Lemma 15 implies that $\kappa(\gamma) - \kappa(\alpha) \leq 3^{-n}$.

This proves (5). If $\overline{\alpha}n \neq \overline{\gamma}n$, then $\kappa(\gamma) - \kappa(\alpha) \geq 3^{-n}$, by Lemma 15. This also proves (5). The case $t < \kappa(\alpha)$ can be treated similarily.

In order to show (4), set $\iota = d_1(t, a') - d_1(t, a)$ and suppose that $\overline{\gamma}n \in B$. Set $\varepsilon = \min(\iota, 3^{-n})$. Fix s with $a \leq s \leq t$. We show that $f(s) \geq \varepsilon$. Note that $d_1(s, C) = s - a$. Fix α. We show that

$$d_2((s, 0), (\kappa(\alpha), \eta_B(\alpha))) - (s - a) \geq \varepsilon.$$

If $a' \leq \kappa(\alpha)$, we obtain

$$d_2((s, 0), (\kappa(\alpha), \eta_B(\alpha))) - (s - a) \geq$$

$$\kappa(\alpha) - s - (s - a) \geq \iota \geq \varepsilon.$$

If $\kappa(\alpha) \leq a$, we obtain

$$d_2((s, 0), (\kappa(\alpha), \eta_B(\alpha))) - (s - a) = s - \kappa(\alpha) + \eta_B(\alpha) - (s - a) =$$

$$\eta_B(\alpha) + a - \kappa(\alpha) \geq 3^{-n} \geq \varepsilon,$$

where $\eta_B(\alpha) + a - \kappa(\alpha) \geq 3^{-n}$ is derived by looking at the cases $\overline{\alpha}n = \overline{\gamma}n$ and $\overline{\alpha}n \neq \overline{\gamma}n$ separately.

Now assume that $f(t) > 0$. We show that γ hits B. If $f(t) > 0$, then $g(t) > t - a$. On the other hand, we have

$$g(t) \leq d_2((t, 0), (a, \eta_B(\gamma))) = t - a + \eta_B(\gamma),$$

so $\eta_B(\gamma) > 0$. By Lemma 16, this implies that γ hits B.

The statement (5) is proved analogously to (4). $\square$

The next lemma is very easy to prove, we just formulate it to be able to refer to it.

Lemma 21. *For real numbers $x < y < z$ and $\delta > 0$ there exists a real number y' such that*

- $x < y' < z$
- $d_1(y, y') < \delta$
- $d_1(x, y') < d_1(y', z)$ *or* $d_1(x, y') > d_1(y', z)$.

For a function F defined on $\{0, 1\}^{\mathbb{N}}$, set

$$F(u) = F(u * 0 * 0 * 0 * \ldots). \tag{6}$$

Now we can show that f has all the desired properties.

Proposition 3. *1. B is a bar $\Leftrightarrow$ f is positive-valued*

2. B is a uniform bar $\Leftrightarrow$ $\inf f > 0$

3. B is co-convex $\Leftrightarrow$ f is weakly convex.

Proof. (1) "$\Rightarrow$". Suppose that B is a bar and fix t. By Proposition 2, we obtain $g(t) > 0$. If $d_1(t, C) < g(t)$, then $f(t) > 0$, by the definition of f. If $0 < d_1(t, C)$, we can apply Lemma 20 to conclude that $f(t) > 0$.

(1) "$\Leftarrow$". If f is positive-valued, then g is positive-valued as well and Proposition 2 implies that B is a bar.

(2) "$\Rightarrow$". If B is a uniform bar, Proposition 2 yields

$$\varepsilon := \inf g > 0.$$

Moreover, there exists n such that $\{0,1\}^n \subseteq B$. Fix $\delta > 0$ such that

$$|s - t| < \delta \;\Rightarrow\; |f(s) - f(t)| < \varepsilon/2$$

for all s and t. Fix t. If $d_1(t, C) < \delta$, we can conclude that

$$f(t) \geq \varepsilon/2$$

by the choice of ε and δ. If $d_1(t, C) > 0$, Lemma 20 and $\{0,1\}^n \subseteq B$ imply that

$$f(t) \geq 3^{-n}.$$

So we have shown that $\inf f \geq \min\left(\varepsilon/2, 3^{-n}\right)$.

(2) "$\Leftarrow$". If $\inf f > 0$, then $\inf g > 0$, and Proposition 2 implies that B is a uniform bar.

(3) "$\Rightarrow$". In view of Remark 5 and Lemma 20, it is sufficient to show that the restriction of f to $-C$ is weakly convex. Fix $t \in -C$ and assume that $f(t) > 0$. Choose a, a', γ and γ' according to Lemma 20. In view of Lemma 21 and the uniform continuity of f, we may assume without loss of generality that either

$$d_1(a, t) < d_1(t, a') \quad \text{or} \quad d_1(a, t) > d_1(t, a').$$

Consider the first case. The second case can be treated analogously. By Lemma 20, we obtain

$$\iota = \inf\{f(s) \mid a \leq s \leq t\} > 0.$$

In particular, $f(\kappa(\gamma)) > 0$, so γ hits B. There exists n such that either

$$\{v \mid v \leq \overline{\gamma}n\} \subseteq B \tag{7}$$

or

$$\{v \mid \overline{\gamma}n \leq v\} \subseteq B\,. \tag{8}$$

Set $\varepsilon = \min\left(\iota, 3^{-n}\right)$. In case (7), we show that

$$\forall s \in -C\,(s \leq t \;\;\Rightarrow\;\; f(s) \geq \varepsilon)\,,$$

as follows. Assume that there exists $s \in -C$ with $s \leq t$ such that $f(s) < \varepsilon$. Then, by the definition of ι, we obtain that $s < a$. Applying Lemma 20 again, we can choose α and α' such that

$$s \in \,]\kappa(\alpha), \kappa(\alpha')[\, \subseteq -C.$$

Then $\overline{\alpha}n \leq \overline{\alpha'}n \leq \overline{\gamma}n$. Thus both $\overline{\alpha}n$ and $\overline{\alpha'}n$ are in B. This implies $f(s) \geq 3^{-n}$, which is a contradiction. In case (8), a similar argument yields

$$\forall s \in -C\,(t \leq s \;\;\Rightarrow\;\; f(s) \geq \varepsilon)\,.$$

(3) "$\Leftarrow$". Assume that f is weakly convex. Fix α and suppose that α hits $\overline{B}$. Then Lemma 19 implies that $f(\kappa(\alpha)) > 0$. There exists n with $\overline{\alpha}n \in B$ such that

$$\forall s \left(s \leq \kappa(\alpha) \;\;\Rightarrow\;\; f(s) > 3^{-n}\right)$$

or

$$\forall s \left(\kappa(\alpha) \leq s \;\;\Rightarrow\;\; f(s) > 3^{-n}\right).$$

Assume the first case. Fix v with $v \leq \overline{\alpha}n$. Then $\kappa(v) \leq \kappa(\alpha)$. If $v \notin B$, then, by Lemma 16 and Lemma 19,

$$f(\kappa(v)) = g(\kappa(v)) \leq \eta_B(v) \leq 3^{-n}.$$

This contradiction shows that

$$\{v \mid v \leq \overline{\alpha}n\} \subseteq B.$$

Now, consider the second case. Fix v with $\overline{\alpha}n < v$. Then $\kappa(\alpha) \leq \kappa(v)$. If $v \notin B$, then $f(\kappa(v)) \leq 3^{-n}$. This contradiction shows that

$$\{v \mid \overline{\alpha}n \leq v\} \subseteq B.$$

$\square$

Part II: Construction of a set B for given f

Set

$$\kappa' : \{0,1\}^{\mathbb{N}} \to [0,1]\,,\ \alpha \mapsto \sum_{k=0}^{\infty} \alpha_k \cdot 2^{-(k+1)}.$$

One cannot prove that κ' is surjective, since this would imply LLPO. Note, however, that every rational $q \in [0,1]$ is in the range of κ'. Moreover, we make use of the following lemma, see [1, Lemma 1].

Lemma 22. *Let S be a subset of $[0,1]$ such that*

$$\forall \alpha\, \exists \varepsilon > 0\, \forall t \in [0,1]\, (|t - \kappa'(\alpha)| < \varepsilon \Rightarrow t \in S)\ .$$

Then $S = [0,1]$.

The next lemma is a typical application of Lemma 22.

Lemma 23. *Fix a uniformly continuous function $f : [0,1] \to \mathbb{R}$ and define*

$$F : \{0,1\}^{\mathbb{N}} \to \mathbb{R},\ \alpha \mapsto f(\kappa'(\alpha)).$$

Then

1. *f is positive-valued $\Leftrightarrow$ F is positive-valued*
2. $\inf f > 0 \;\Leftrightarrow\; \inf F > 0.$

Proof. In (1), the direction "$\Rightarrow$" is clear. For "$\Leftarrow$", apply Lemma 22 to the set

$$S = \{t \in [0,1] \mid f(t) > 0\}\,.$$

The equivalence (2) follows from the density of the image of κ' in $[0,1]$ and the uniform continuity of f. □

In the following proposition, we use a similar construction as in [2].

Proposition 4. *For every uniformly continuous function*

$$f : [0,1] \to \mathbb{R}$$

there exists a detachable subset B of $\{0,1\}^*$ which is closed under extension such that

1. *B is a bar $\Leftrightarrow$ f is positive-valued*
2. *B is a uniform bar $\Leftrightarrow$ $\inf f > 0$*

3. B is co-convex $\Leftrightarrow$ *f is weakly convex.*

Proof. Since the function

$$F : \{0,1\}^{\mathbb{N}} \to \mathbb{R},\ \alpha \mapsto f(\kappa'(\alpha))$$

is uniformly continuous, there exists a strictly increasing function $M : \mathbb{N} \to \mathbb{N}$ such that

$$|F(\alpha) - F(\overline{\alpha}(M(n)))| < 2^{-n}$$

for all α and n, recalling the convention given in (6). Since M is strictly increasing, for every k the statement

$$\exists n\,(k = M(n))$$

is decidable. Therefore, for every u we can choose $\lambda_u \in \{0,1\}$ such that

$$\begin{aligned}
\lambda_u = 0 \quad &\Rightarrow \quad \forall n\,(|u| \neq M(n)) \;\vee\; \exists n\,\big(|u| = M(n) \;\wedge\; F(u) < 2^{-n+2}\big) \\
\lambda_u = 1 \quad &\Rightarrow \quad \exists n\,\big(|u| = M(n) \;\wedge\; F(u) > 2^{-n+1}\big)\,.
\end{aligned}$$

The set

$$B = \{u \in \{0,1\}^* \mid \exists l \leq |u|\,(\lambda_{\overline{u}l} = 1)\}$$

is detachable and closed under extension. Note that

$$F(\alpha) \geq 2^{-n+3} \;\Rightarrow\; \overline{\alpha}(M(n)) \in B \tag{9}$$

and

$$\overline{\alpha}(M(n)) \in B \;\Rightarrow\; F(\alpha) \geq 2^{-n} \tag{10}$$

for all α and n. In view of Lemma 23, (9) and (10) yield (1) and (2).

In order to show (3), fix a co-convex set B. Moreover, fix $t \in [0,1]$ and assume that $f(t) > 0$. By Remark 5, we may assume that t is a rational number, which implies that there exists α such that $\kappa'(\alpha) = t$. Now $F(\alpha) > 0$ implies that α hits B. Therefore, there exists n such that either

$$\{v \mid v \leq \overline{\alpha}n\} \subseteq B$$

or

$$\{v \mid \overline{\alpha}n \leq v\} \subseteq B.$$

In the first case, we show that

$$\inf\{f(s) \mid s \in [0,t]\} \geq \min\big(2^{-n}, F(\alpha)\big)\,. \tag{11}$$

Assume that there exists $s \leq t$ such that $f(s) < 2^{-n}$ and $f(s) < F(\alpha)$. The latter implies that $s < t$. Choose a β with the property that $\kappa'(\beta)$ is close enough to s such that

$$\kappa'(\beta) < \kappa'(\alpha) \tag{12}$$

and

$$F(\beta) = f(\kappa'(\beta)) < 2^{-n}. \tag{13}$$

Now (10) and (13) imply that $\overline{\beta}n \notin B$. On the other hand, (12) implies that $\overline{\beta}n \leq \overline{\alpha}n$ and therefore $\overline{\beta}n \in B$. This is a contradiction, so we have shown (11).

In the case

$$\{v \mid \overline{\alpha}n \leq v\} \subseteq B$$

we can similarly show that

$$\inf \{f(s) \mid s \in [t, 1]\} \geq \min \left(2^{-n}, F(\alpha)\right).$$

Now assume that f is weakly convex. Fix an α which hits B. Then there exists n with $\overline{\alpha}(M(n)) \in B$ and (10) implies that $f(\kappa'(\alpha)) > 0$. We choose n large enough such that either

$$\inf \{f(t) \mid t \in [0, \kappa'(\alpha)]\} \geq 2^{-n+3}$$

or

$$\inf \{f(t) \mid t \in [\kappa'(\alpha), 1]\} \geq 2^{-n+3}.$$

By (9), we obtain

$$\{v \mid v \leq \overline{\alpha}(M(n))\} \subseteq B$$

in the first case and

$$\{v \mid \overline{\alpha}(M(n)) \leq v\} \subseteq B.$$

in the second. Therefore, B is co-convex.

□

Thus the proof of Theorem 6 is completed. We conclude this section with a discussion about weakly convex functions.

Remark. *Uniformly continuous, (quasi-)convex functions $f : [0, 1] \to \mathbb{R}$ are weakly convex. To this end, we recall that f is* convex *if we have*

$$f(\lambda s + (1 - \lambda)t) \leq \lambda f(s) + (1 - \lambda) f(t)$$

and quasiconvex *if we have*

$$f(\lambda s + (1 - \lambda)t) \leq \max (f(s), f(t))$$

for all $s, t \in [0,1]$ *and all* $\lambda \in [0,1]$. *Clearly, convexity implies quasi-convexity. Now assume that* f *is quasi-convex. Fix* $t \in [0,1]$ *and assume that* $f(t) > 0$. *Set* $\varepsilon = f(t)/2$. *The assumption that both*

$$\inf\{f(s) \mid s \in [0,t]\} < f(t) \quad \textit{and} \quad \inf\{f(s) \mid s \in [t,1]\} < f(t)$$

is absurd, because in that case by uniform continuity there exists $s < t < s'$ *such that* $f(s) < f(t)$ *and* $f(s') < f(t)$. *Compute* $\lambda \in (0,1)$ *such that* $t = \lambda s + (1-\lambda)s'$, *and note that quasi-convexity of* f *implies* $f(t) \leq \max(f(s), f(s')) < f(t)$ *which is absurd. Hence, either* $\inf\{f(s) \mid s \in [0,t]\} > \varepsilon$ *or* $\inf\{f(s) \mid s \in [t,1]\} > \varepsilon$.

Pointwise continuous functions on $[0,1]$ *which are weakly decreasing on* $[0,s]$ *and weakly increasing on* $[s,1]$ *for some* s *are weakly convex. See [10] for a detailed discussion of various notions of convexity.*

If f *is weakly convex, then the set* $\{t \mid f(t) \leq 0\}$ *is convex. With classical logic, the reverse implication holds as well, if* f *is continuous. This illustrates that weak convexity is indeed a convexity property.*

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