Optimal Risk Sharing in Insurance Networks
An Application to Asset-Liability Management

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Abstract

We discuss the impact of risk sharing and asset-liability management on capital requirements. Our analysis contributes to the evaluation of the merits and deficiencies of different risk measures. In particular, we highlight that the class of V@R-based risk measures allows for a substantial reduction of the total capital requirement in corporate networks that share risks between entities. We provide case studies that complement previous theoretical results and demonstrate their practical relevance. For large networks, optimal asset-liability management is often contrary to those strategies that are desirable from a regulatory point of view.

Keywords: Corporate Networks, Optimal Risk Sharing, Distortion Risk Measures, Value at Risk, Average Value at Risk, Range Value at Risk, Solvency Capital Requirement.

1 Introduction

Capital requirements of insurance companies or banks serve as a protection of policy holders, banking customers, and creditors. They provide a buffer against downside risk, i. e., the adverse random fluctuations of the financial resources of a company. In internal models of financial institutions such capital requirements are computed on the basis of the simulated distribution of the firm's book value of equity at a finite time horizon, also called the future net asset value. The resulting capital requirements depend on the risk measures that are used. While the regulation scheme Solvency II is based on the risk measure value at risk (V@R), the Swiss Solvency Test employs a coherent risk measure, average value at risk (AV@R), also known as expected shortfall. The influence of risk measures on capital requirements as well as their properties have been the subject of intense scientific research over the last twenty years, see, e. g., Föllmer & Schied (2016) or Föllmer & Weber (2015).

In this paper, we discuss the impact of risk sharing and asset-liability management on capital requirements. This investigation will contribute to the evaluation of the merits and deficiencies of different risk measures. In particular, we highlight that the class of V@R-based risk measures, as defined in Weber (2018), allows for a substantial reduction of the total capital requirement in corporate networks that share risks between entities. We provide case studies that complement the theoretical analysis of Embrechts, Liu & Wang (2018) and Weber (2018) and illustrate their practical relevance. In addition, we refine in Section 2.2.2 the tail allocations suggested in these papers to ensure that downside risk is shared in an approximately symmetric manner. Such an analysis of optimal risk sharing within a model for asset-liability management is – to the best of our knowledge – new to the literature.

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The paper is structured as follows: Section 2.1 first reviews the notion of solvency capital requirements, emphasizing that the definition commonly used in practice deviates from an alternative definition that is naturally derived from the notion of acceptability in the theory of monetary risk measures. Second, in Section 2.2, we consider risk sharing between entities. If networks of companies are not required to report a total capital requirement on the basis of a consolidated balance sheet, risk sharing may serve as an instrument to reduce required capital. We review the general risk sharing problem and the notion of inf-convolutions, summarize theoretical results for distortion risk measures of Embrechts et al. (2018) or Weber (2018) and refine the allocations of the distribution's tail within the network. Section 3 introduces a model setting that admits the joint analysis of asset-liability management and risk sharing. The general structure is described in Section 3.1.1. In the context of a Black-Scholes asset model and for deterministic liabilities, the inf-convolutions and capital requirements are explicitly computed for three important examples: average value at risk and the two V@R-type risk measures value at risk and range value at risk. More sophisticated models are then analyzed on the basis of Monte Carlo case studies. Section 3.2 describes how distributions and parameters are chosen, and how we calibrate value at risk, average value at risk and range value at risk in order to allow for a meaningful comparison of these risk measures. In Section 3.3, we analyze three case studies of different complexity: a) Assets are modeled by a Black-Scholes market, liabilities are deterministic. b) Liabilities may be random; different types of dependence between assets and liabilities are investigated. c) An additional left-tailed asset is available. We find that corporate networks may largely hide downside risk, if capital requirements are computed on the basis of V@R-type risk measures. For large networks, optimal asset-liability management is often contrary to those strategies that are desirable from a regulatory point of view. The results are quite striking, and thus we discuss this issue in detail.

Literature. Risk sharing constitutes a main principle in actuarial risk theory and has been studied in a wide variety of settings, starting with the pioneering work of Borch (1962), see, e. g., also Wilson (1968) and Raviv (1979) for seminal contributions. The general problem of optimal risk sharing is still an ongoing field of research.

Barrieu & El Karoui (2005) and Barrieu & El Karoui (2008) introduced the inf-convolution in order to formulate the risk sharing problem among agents with *convex* risk measures. They show that the inf-convolution of two convex risk measures is again a convex risk measure. The optimal structure of the minimization problem is explicitly derived when agents have dilated risk measures, i. e., $\rho_\gamma(Z) = \frac{1}{\gamma}\,\rho\,(\gamma Z)$. Jouini, Schachermayer & Touzi (2008) show that for distribution-based, also called law-invariant, concave monetary utility functions the set of Pareto optimal comonotone allocations is non-empty. Acciaio (2007) considers monetary functionals that are not necessarily monotone and characterizes optimal solutions and their existence. The paper introduces best monotone approximations of non-monotone functionals where the resulting optimization corresponds to the inf-convolution with constraint $\sum_{i=1}^n Z_i \leq Z$ defined by Filipovic & Kupper (2008) and Filipovic & Svindland (2008). Explicit calculations of optimal risk sharing rules for particular cases are given. Further risk sharing strategies for special cases of two or three agents can be found in Acciaio (2005). The case of distribution-based and cash-invariant convex functions that are not necessarily monotone is also considered by Filipovic & Svindland (2008); the authors prove that the capital and risk allocation problem always admits a solution via contracts whose payoffs are defined as increasing Lipschitz-continuous functions of the total risk. Ludkovski & Young (2009) study optimal risk sharing among n agents endowed with convex distortion risk measures and determine Pareto optimal allocations. For convex distortion risk measures all Pareto optimal redistributions between firms and competitive equilibria are characterized by Boonen (2015) in the context of finite scenario spaces.

In contrast to most results in the literature, in the current paper we consider risk measures that are not necessarily convex. This direction is also taken by the following papers: Risk sharing for V@R is considered in Galchion (2010). For V@R and AV@R, Asimit, Badescu & Tsanakas (2013) derive optimal risk transfers within insurance groups that minimize the risk adjusted value of the group liabilities when valuation takes place under a cost-of-capital methodology. Embrechts, Liu & Wang (2018) solve the optimal risk sharing problem for value at risk and range value at risk and state robustness results for optimal allocations. Weber (2018) solves the risk sharing problem for general distortion risk measures that are not necessarily convex and applies these results to corporate networks, for example, to insurance networks. The paper includes a discussion of the difference of the notions *network* and *group*. Weber's (2018) framework includes the results of Embrechts et al. (2018) as special cases. We review these results in Section 2.2 – extending them by a construction of tail allocations in the face of the issue of fairness.

2 Capital Regulation and Network Risk Minimization

2.1 Capital Requirements

Capital requirements are a cornerstone of regulation schemes such as *Basel III* for banks, *Solvency II* for European insurance companies, or the *Swiss Solvency Test* for insurance companies in Switzerland. The key idea is that financial firms should hold a buffer for potential losses that ensures the firm's financial solvability and thereby serves to protect customers, policy holders and other counterparties. The computation of such a capital requirement - in the sequel named *solvency capital requirement* (SCR) - typically involves two components, *stochastic balance sheet projections* capturing the random evolution of the firm's equity over a given time horizon, and a *monetary risk measure* that quantifies the inherent risk on a monetary scale or, equivalently, specifies acceptability of financial positions, e. g., from the perspective of a financial supervisory authority, a rating agency, or the board of management.

To formalize the SCR computation in a stylized manner, let us consider an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a one period economy with two dates, say $t = 0, 1$. Time 0 is interpreted as today, time 1 as the future time horizon of the regulation scheme, e. g., one year in case of Solvency II. We denote by $\mathcal{X}$ the set of financial positions at time 1 whose risk needs to be assessed. By sign convention negative values correspond to debt or losses. Throughout this paper, $\mathcal{X}$ is a vector space of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ that contains the constants.

Assets and Liabilities. At time $t = 0, 1$, the economic values of assets and liabilities of a financial firm according to the solvency balance sheet are denoted by A_t and L_t, respectively, and the book value of equity or net asset value (NAV) is then derived as $E_t = A_t - L_t$. Note that the quantities A_0, L_0, E_0 at $t = 0$ are deterministic, while their counterparts A_1, L_1, E_1 at $t = 1$ are typically not known in advance, but random. Mathematically, the values of assets and liabilities A_1, L_1 and the resulting equity E_1 are modeled as real-valued random variables on the given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In practice, these quantities can be derived from stochastic balance sheet projections within sophisticated *internal models* that rely extensively on Monte Carlo simulations.

Solvency Capital Requirement. Regulatory guidelines typically describe requirements on the SCR computation verbally, but do not provide an exact and unique SCR definition in mathematical terms. In particular, as illustrated in Example 2.1 below for Solvency II regulation, regulatory requirements can be contradictory, leaving considerable room for interpretation.

In this paper, we focus on two different SCR definitions:

$$\mathrm{SCR}_{\mathcal{A}}(E_1) := \rho(E_1 - E_0),$$
$$\mathrm{SCR}_{\mathrm{mean}}(E_1) := \rho(E_1 - \mathbb{E}[E_1]),$$

where ρ denotes a monetary risk measure with acceptance set $\mathcal{A}$ such as *value at risk* (V@R), *average value at risk* (AV@R) or *range value at risk* (RV@R), see Section A for a short review.

While $\mathrm{SCR}_{\mathcal{A}}(E_1)$ evaluates the risk of the random capital increment $E_1 - E_0$ over the given time horizon (neglecting discounting effects on the one-year horizon), the alternative definition $\mathrm{SCR}_{\mathrm{mean}}(E_1)$ refers to the firm's centered equity $E_1 - \mathbb{E}[E_1]$ at time 1. Also note that

$$\mathrm{SCR}_{\mathcal{A}}(E_1) = E_0 + \rho(E_1) \quad \text{and} \quad \mathrm{SCR}_{\mathrm{mean}}(E_1) = \mathbb{E}[E_1] + \rho(E_1),$$

due to cash-invariance of the monetary risk measure ρ.

From a conceptional point of view, the definition $\mathrm{SCR}_{\mathcal{A}}$ corresponds to a regulator's perspective, and it is based on the natural requirement that equity E_1 at time 1 should be acceptable with respect to a prescribed monetary risk measure ρ, i. e.,

$$E_1 \in \mathcal{A} \quad \Leftrightarrow \quad \rho(E_1) \leq 0.$$

For $\mathrm{SCR}_{\mathcal{A}}(E_1) = E_0 + \rho(E_1)$, acceptability of the firm's equity E_1 is equivalent to $\mathrm{SCR}_{\mathcal{A}}(E_1) \leq E_0$, i. e., the firm's equity is sufficient to cover the solvency capital requirement. In practice, however, it is a common approach to consider only unexpected losses, in particular for market risks and underwriting risks. This leads to the alternative definition $\mathrm{SCR}_{\mathrm{mean}}$.

Example 2.1. For Solvency II regulation, Recital 64 of the Directive 2009/138/EC states that capital must be sufficient to prevent ruin with probability 99.5% on a one-year time horizon, i. e., $\mathbb{P}[E_1 < 0] \leq \alpha$ with $\alpha = 0.005$. This condition is equivalent to $E_1 \in \mathcal{A}_{\mathrm{V@R}_{0.005}}$, where $\mathcal{A}_{\mathrm{V@R}_{0.005}}$ denotes the acceptance set of value at risk defined in (12). Hence, a canonical SCR definition in the context of Solvency II is

$$\mathrm{SCR}_{\mathcal{A}}(E_1) := \mathrm{V@R}_{0.005}(E_1 - E_0) = E_0 + \mathrm{V@R}_{0.005}(E_1) = E_0 - q^{+}_{E_1}(0.005),$$

where $q^{+}_{E_1}$ denotes the upper quantile function of E_1.

Contradicting, §101(2) of the Directive 2009/138/EC prescribes that the SCR "shall cover only unexpected losses", and that "it shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99.5 % over a one-year period." This supports the definition in terms of the so-called *mean value at risk*

$$\mathrm{SCR}_{\mathrm{mean}}(E_1) := \mathrm{V@R}_{0.005}(E_1 - \mathbb{E}[E_1]) = \mathbb{E}[E_1] + \mathrm{V@R}_{0.005}(E_1) = \mathbb{E}[E_1] - q^{+}_{E_1}(0.005)$$

which is widely used in practice. Both definitions are consistent to specific regulatory requirements, but lead however to different solvency capital requirements.

Financial institutions are typically owned by shareholders with limited liability. The free surplus - given as equity less SCR - can be distributed as dividends to the shareholders. Consequently, shareholders and the management board are interested in reducing the SCR via appropriate techniques. In the sequel, we focus on this problem from a network's perspective.

2.2 The Risk Sharing Problem of the Network

2.2.1 Inf-Convolutions

Consider a financial network that consists of n entities that are all individually subject to capital regulation. We suppose that the solvency capital requirement of entity $i = 1, 2, \ldots, n$ is computed based on a monetary risk measure ρ^i, and we write $\mathrm{SCR}^i_{\mathcal{A}}$ and $\mathrm{SCR}^i_{\mathrm{mean}}$, respectively, to differentiate between the two different SCR definitions for entity i.

The network's balance sheet is obtained by consolidating the individual balance sheets of its sub-entities. Denoting from now on by A_t and L_t the total consolidated assets and liabilities at times $t = 0, 1$, the network's total equity is given by $E_t = A_t - L_t$, $t = 0, 1$. The corporate network now uses at time $t = 0$ legally binding transfer agreements to modify the equities at time $t = 1$.

The resulting new allocation is denoted by $(E^i)_{i=1,\dots,n}$, where $\sum_{i=1}^n E_1^i = E_1$ and $\sum_{i=1}^n E_0^i = E_0$. In this situation, the total SCR of the network is given by

$$\sum_{i=1}^n \mathrm{SCR}_{\mathcal{A}}^i(E_1^i) = E_0 + \sum_{i=1}^n \rho^i(E_1^i) \quad \text{and} \quad \sum_{i=1}^n \mathrm{SCR}_{\mathrm{mean}}^i(E_1^i) = \mathbb{E}[E_1] + \sum_{i=1}^n \rho^i(E_1^i), \quad \text{respectively.}$$

This definition relies on the assumption that the firm's individual SCRs are added up to obtain the network's SCR. In particular, this means that the network's SCR is not computed based on a consolidated solvency balance sheet.

For both SCR definitions, the minimization of the network's SCR is equivalent to the minimization of $\sum_{i=1}^n \rho^i\left(E_1^i\right)$. In other words, for a fixed number of n firms the problem of the network consists in the design of optimal transfers that minimize $\sum_{i=1}^n \rho^i\left(E_1^i\right)$. We thus face the optimal risk sharing problem

$$\square_{i=1}^n \rho^i(E_1) = \inf\left\{ \sum_{i=1}^n \rho^i\left(E_1^i\right) \,\middle|\, \sum_{i=1}^n E_1^i = E_1,\ E_1^1,\dots,E_1^n \in \mathcal{X} \right\}, \tag{1}$$

also known as *inf-convolution*. Let us write

$$\square_{i=1}^n \mathrm{SCR}_{\mathcal{A}}^i(E_1) = E_0 + \square_{i=1}^n \rho^i(E_1) \quad \text{and} \quad \square_{i=1}^n \mathrm{SCR}_{\mathrm{mean}}^i(E_1) = \mathbb{E}[E_1] + \square_{i=1}^n \rho^i(E_1) \tag{2}$$

for the corresponding solvency capital requirements.

Remark 2.2. Let ρ be a coherent risk measure and assume that $\rho^i = \rho$ for any firm $i = 1,\dots,n$, $n \in \mathbb{N}$. In this case, optimal risk sharing and splitting the risk within the network to more firms do not reduce the total network's risk, i. e.,

$$\square_{i=1}^n \rho(E_1) = \rho(E_1) \quad \text{for all } n \in \mathbb{N}.$$

Indeed, for all decompositions $E_1 = E_1^1 + \ldots + E_1^n$, subadditivity yields

$$\rho(E_1) = \rho\left(\sum_{i=1}^n E_1^i\right) \le \sum_{i=1}^n \rho(E_1^i),$$

and this lower bound is attained for $E_1^i = \alpha^i E_1$, $i = 1,\dots,n$, with $\alpha^1 + \ldots + \alpha^n = 1$. In particular, it is optimal to allocate the total net asset value to one entity, e. g., the holding company.

2.2.2 Risk Sharing with Distortion Risk Measures

In the context of distortion risk measures, problem (1) is discussed in Weber (2018). The risk measures V@R, AV@R and RV@R belong to this class of risk measures. Theorem 2.4 in Weber (2018) provides an upper bound to the solution and an allocation that attains this bound. The results characterize under which conditions the bound is attained and generalize the work of Embrechts et al. (2018).

Definition 2.3. An increasing function $g : [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) = 1$ is called a *distortion function*. If $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$, then

$$c^g(A) := g(\mathbb{P}[A]), \quad A \in \mathcal{F},$$

defines a capacity. The risk measure

$$\rho^g(X) := \int (-X)\, dc^g,$$

defined as the Choquet integral with respect to c^g, is called *distortion risk measure*.

As special cases, Weber (2018) introduces the class of V@R-type distortion risk measures.

Definition 2.4. Consider the class of distortion functions g such that

$$g(x) = 0, \quad \forall x \in [0, \alpha]$$
$$g(x) > 0, \quad \forall x \in (\alpha, 1]$$

for some $\alpha \in [0, 1)$. The number α is called the *parameter* of g, and

$$\hat{g}(x) = \begin{cases} g(x+\alpha), & 0 \le x \le 1-\alpha \\ 1, & 1-\alpha < x \end{cases}$$

is the *active part* of g. If the parameter $\alpha > 0$, then ρ^g is called a V@R-*type distortion risk measure.*

For V@R-type risk measures, the alternative representation $\rho^g(X) = \int_{[0,1]} \mathrm{V@R}_\lambda(X)\, g(d\lambda)$ of distortion risk measures as mixtures of value at risk takes the form

$$\rho^g(X) = \int_{[\alpha,1]} \mathrm{V@R}_\lambda(X)\, g(d\lambda),$$

i. e., a risk measure of V@R-type does not depend on any properties of the tail of X beyond its V@R at level α. The risk measures V@R and RV@R are of V@R-type, AV@R is not. This is shown in Table 1.

Risk Measure	$\mathrm{V@R}_\alpha$	$\mathrm{AV@R}_\beta$	$\mathrm{RV@R}_{\alpha,\beta}$
g(x) =	$\begin{cases} 0, & 0 \le x \le \alpha \\ 1, & \alpha < x \end{cases}$	$\begin{cases} \frac{x}{\beta}, & 0 \le x \le \beta \\ 1, & \beta < x \end{cases}$	$\begin{cases} 0, & 0 \le x \le \alpha \\ \frac{x-\alpha}{\beta}, & \alpha < x \le \alpha+\beta \\ 1, & \alpha+\beta < x \end{cases}$
Type	V@R-type	**Not** V@R-type	V@R-type

Table 1: Distortion functions for the risk measures V@R, AV@R and RV@R for $\alpha, \beta > 0$ with $\alpha + \beta \le 1$.

The solution to the optimal risk sharing problem (1) minimizes the network's total risk. The minimizer is an allocation $\left(E^i\right)_{i=1,\dots,n}$ with $\sum_{i=1}^n E_1^i = E_1$ and $\sum_{i=1}^n E_0^i = E_0$. The next paragraph describes – as a self-contained presentation – the structure of the solution derived in Weber (2018), Theorem 2.4. It provides a translation of the results, formulated in Weber (2018) in terms of losses L, to financial positions $X = -L$ and prepares the subsequent discussion of the new results on tail allocation.

Basis Results on Optimal Risk Sharing. Let $E_1 \in L^\infty$ and $n \in \mathbb{N}$. By $g^1, g^2, \dots, g^n$ we denote left-continuous distortion functions with parameters $\alpha_1, \alpha_2, \dots, \alpha_n \in [0, 1)$ and define $d = \sum_{i=1}^n \alpha_i$. We set $\rho^i = \rho^{g_i}$, i. e., ρ^i is the distortion risk measure associated with the distortion function g^i, $i = 1, 2, \dots, n$. Define the left-continuous functions

$$f = \min\left\{\hat{g}^1, \hat{g}^2, \dots, \hat{g}^n\right\}, \qquad g(x) = \begin{cases} 0, & 0 \le x \le d \wedge 1, \\ f(x-d), & d \wedge 1 < x \le 1 \end{cases}$$

Note that $g \equiv 0$, if $d \ge 1$. In particular, g is not necessarily a distortion function with $g(1) = 1$. We set $\mathrm{V@R}_\lambda := \mathrm{V@R}_1 = -\operatorname{ess\,sup}$ for $\lambda \ge 1$.

1. There exist $E_1^1, E_1^2, \dots, E_1^n \in L^\infty$ such that $\sum_{i=1}^n E_1^i = E_1$ and

$$\sum_{i=1}^n \rho^i(E_1^i) = \int_{[0,1]} \mathrm{V@R}_\lambda(E_1) g(d\lambda) + (g(1) - 1) \operatorname{ess\,sup} E_1.$$

If $d \ge 1$, this equation can be simplified and we obtain

$$\sum_{i=1}^n \rho^i(E_1^i) = -\operatorname{ess\,sup} E_1.$$

2. The allocation $(E_1^i)_{i=1,2,\dots,n}$ can be constructed as follows. Let

$$Y := E_1 - \operatorname{ess\,sup} E_1 \ \le\ 0.$$

There exists a random variable U, uniformly distributed on $[0,1]$, such that $Y = -\mathrm{V@R}_U(Y)$. For $i = 1,2,\dots,n$, we set

$$r_i(\lambda) = \begin{cases} 1, & i = \inf\{j : \hat{g}_j(1-\lambda) = f(1-\lambda)\}, \\ 0, & \text{else}, \end{cases}$$

($\lambda \in [0,1]$) and $R_i(y) = -\int_0^{|y|} r_i(\lambda)d\lambda$. We define $\tilde{Y} = Y \cdot \mathbf{1}_{\{U \ge d\}}$ and $\tilde{E}_1^i = R_i(\tilde{Y})$. For $i = 1,2,\dots,n$, we set

$$E_1^i \quad = \quad Y \cdot \mathbf{1}_{\left\{\sum_{l=1}^{i-1} \alpha_l \,\le\, U \,<\, \sum_{l=1}^{i} \alpha_l\right\}} \ + \ \tilde{E}_1^i \ + \ \frac{\operatorname{ess\,sup} E_1}{n} \tag{3}$$

If $d \ge 1$, this equation can be simplified and we obtain

$$E_1^i = Y \cdot \mathbf{1}_{\left\{\sum_{l=1}^{i-1} \alpha_l \,\le\, U \,<\, \sum_{l=1}^{i} \alpha_l\right\}} \ + \ \frac{\operatorname{ess\,sup} E_1}{n}$$

These results can be generalized to unbounded random variables, see Weber (2018).

Tail Allocation. The preceding paragraph characterizes a particular solution to (1), but for V@R-based risk measures multiple solutions are admissible. V@R-based risk measures ignore the extreme tail. This implies that the tail part of the distribution of E_1 that is hidden via risk sharing can be allocated to different entities in various ways. While V@R-based risk measurements remain invariant under these re-allocations of the tail, other quantities that are important from the perspective of the single entities may change, e. g., the profit of the individual firms in the network. In contrast to Embrechts et al. (2018) and Weber (2018), we construct alternative tail allocations; these do also minimize the network's total risk, but may provide fairer allocations of the extreme downside risk from the perspective of the single firms.

Remark 2.5. In Eq. (3), the terms

$$Y \cdot \mathbf{1}_{\left\{\sum_{l=1}^{i-1} \alpha_l \,\le\, U \,<\, \sum_{l=1}^{i} \alpha_l\right\}} \tag{4}$$

refer to a particular example of an allocation of the extreme downside of Y resp. E_1 to the different entities. This part of the distribution becomes invisible in the risk measurement $\sum_{i=1}^n \rho^i(E_1^i)$: *It is swept under the carpet!* An allocation according to (4) does, however, not symmetrically share extreme downside among entities. More generally, the terms (4) can be replaced by

$$\check{\mathrm{E}}_1^i := Y \cdot s_i(U) \tag{5}$$

for càdlàg functions $s_i : [0,d] \to \{0,1\}$, $\int_0^d s_i(u)du = \alpha_i$, $i = 1,2,\dots,n$, with $\sum_{i=1}^n s_i \equiv 1$. With this modification, an alternative solution to problem (1) is

$$E_1^i = \check{\mathrm{E}}_1^i + \tilde{E}_1^i + \frac{c}{n} \tag{6}$$

where $c := \operatorname{ess\,sup} E_1$.

Let us study the extreme downside risk in more detail. For simplicity, we assume that Y has a continuous distribution. Since $Y = -\mathrm{V@R}_U(Y)$, the random variables Y and U are comonotone. Extreme downside risk up to probability d corresponds to the event:

$$C := \{E_1 < -\mathrm{V@R}_d(E_1)\} = \{Y < -\mathrm{V@R}_d(Y)\} = \{U < d\}.$$

Setting $y_d := -\mathrm{V@R}_d(Y)$, the allocation (6) can be characterized more specifically by

$$E_1^i \;=\; \begin{cases} \check{E}_1^i + \frac{c}{n} \;<\; y_d + \frac{c}{n} & \text{on } C \\ \tilde{E}_1^i + \frac{c}{n} \;\geq\; y_d + \frac{c}{n} & \text{on } C^c \end{cases}$$

with $\mathbb{P}[E_1^i < y_d + \frac{c}{n}] = \alpha_i$, $i = 1, 2, \ldots, n$. Setting $C^i := \{E_1^i < y_d + \frac{c}{n}\}$, $i = 1, 2, \ldots, n$, $(C^i)_{i=1,2,\ldots,n}$ defines a partition of C, and C^i corresponds to the set of scenarios in which the downside risk is allocated to entity i. Fairness of an allocation of the downside risk can be interpreted in terms of the probabilities $\mathbb{P}[C_i] = \alpha_i$ and in terms of the conditional distribution of the severity of losses, i. e., $\mu^i(\cdot) := \mathbb{P}[E_1^i \in \cdot | C^i]$. This probability measure, μ^i, is equal to the distribution of $-\mathrm{V@R}_{U^i}(Y) + c/n$ for a random variable U^i with distribution function

$$F^i(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{\alpha_i} \int_0^x s_i(u) du & \text{if } 0 \leq x < d \\ 1 & \text{if } x \geq d \end{cases}$$

Let us assume that $(\alpha_i)_{i=1,2,\ldots,n}$ are fixed. We consider the issue of fairness in terms of the conditional distributions of the severity of losses. To be precise, we provide a sequence of allocations of the extreme downside as specified in (5) such that the conditional distributions of the severity of losses are asymptotically equal. In this sense, the sequence of allocations becomes "fair in the limit". An example is the sequence

$$s_i^m = \begin{cases} 1, & \frac{k}{m} \cdot d + \frac{\sum_{l=1}^{i-1} \alpha_l}{m} \;\leq\; x \;<\; \frac{k}{m} \cdot d + \frac{\sum_{l=1}^{i} \alpha_l}{m} \text{ for some } k = 0, 1, \ldots, m-1, \\ 0, & \text{else.} \end{cases}$$

Denoting the corresponding conditional severity distributions by $\mu^{i,m}$, $i = 1, 2, \ldots, n$, $m \in \mathbb{N}$, the sequence $(\mu^{i,m})_{m \in \mathbb{N}}$ converges weakly as $m \to \infty$ to a probability measure μ^∞, for each i. The measure μ^∞ does not depend on i and equals the distribution of $-\mathrm{V@R}_{U^\infty}(Y) + c/n$ for a random variable U^∞ with distribution function

$$F^\infty(x) \;=\; \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{d} & \text{if } 0 \leq x < d \\ 1 & \text{if } x \geq d \end{cases}$$

Special Cases. For the particular distortion risk measures V@R, AV@R and RV@R, we recover the results in Embrechts et al. (2018), Theorem 2.

Example 2.6. For any $E_1 \in \mathcal{X}$ and $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \geq 0$, $n \in \mathbb{N}$, we have

(a) $\square_{i=1}^n \mathrm{V@R}_{\alpha_i}(E_1) = \mathrm{V@R}_{\sum_{i=1}^n \alpha_i}(E_1)$,

(b) $\square_{i=1}^n \mathrm{AV@R}_{\beta_i}(E_1) = \mathrm{AV@R}_{\max\{\beta_1,\ldots,\beta_n\}}(E_1)$,

(c) $\square_{i=1}^n \mathrm{RV@R}_{\alpha_i,\beta_i}(E_1) = \mathrm{RV@R}_{\sum_{i=1}^n \alpha_i, \max\{\beta_1,\ldots,\beta_n\}}(E_1)$.

Note that the optimal risk sharing problem (1) can be combined with other management actions. For example, the network may adjust its structure by increasing the number of firms over longer time horizons, or the network may optimize its asset allocation to further reduce its total risk and its total SCR (cf. Section 3). In particular, Weber (2018) shows that for V@R-type risk measures and sufficiently large n, the corporate network can find a capital allocation such that

$$\square_{i=1}^n \rho^i(E_1) = -\operatorname{ess\,sup} E_1, \tag{7}$$

corresponding to the best case scenario. Downside risk can thus completely be hidden within corporate network structures.

V@R is a special case of a V@R-type distortion risk measure, and hence our observations are relevant in the context of Solvency II. In contrast, they do not apply to the Swiss Solvency Test that uses the coherent risk measure AV@R as the basis for capital regulation (cf. Remark 2.2).

3 An Application to Asset-Liability Management

This section provides numerical case studies on optimal risk sharing. In Section 3.1, we introduce an asset-liability management (ALM) model for networks. Entities can implement various (static) asset allocation strategies over a one-year time horizon. Within this framework we analyze three case studies of different complexity:

1. Assets are modeled by a Black-Scholes market, liabilities are deterministic.
2. Liabilities may be random; different types of dependence between assets and liabilities are investigated.
3. An additional left-tailed asset is available.

For these cases, we quantify the impact of the number n of sub-entities in the network on the network's minimal risk $\square_{i=1}^n \rho^i(E_1)$ and on the solvency capital requirements $\square_{i=1}^n \mathrm{SCR}^i_{\mathcal{A}}(E_1)$ and $\square_{i=1}^n \mathrm{SCR}^i_{\mathrm{mean}}(E_1)$. We demonstrate how asset-liability management can further reduce the minimal network risk. We focus on three different risk measures: V@R, AV@R and RV@R.

3.1 Asset-Liability Management Model

3.1.1 General Asset-Liability Model

Consider an ALM model with finite time horizon 1. We assume that the network's firms can invest in a financial market with a finite number $K \geq 1$ of liquidly traded assets. We denote by S^k_t the price of one share of asset $k = 1, \dots, K$, and by L_t the consolidated liabilities at time $t \in [0,1]$, respectively. At $t = 0$ the network decides – in a static manner – how to invest in the different assets in the period $t \in [0,1]$ by determining an *asset allocation strategy* $\delta \in \mathbb{R}^K$ with

$$\delta^k \geq 0 \quad \text{and} \quad \sum_{k=1}^K \delta^k = 1,$$

where δ^k denotes the fraction of the total asset amount of the balance sheet invested in asset k. The corresponding numbers of shares held in the assets $k = 1, \dots, K$ are given by

$$\eta^k(\delta) = \delta^k \cdot \frac{E_0 + L_0}{S^k_0},$$

where E_0 is the net asset value – or book value of equity – at time 0. Afterwards the net asset value, calculated as the difference of total assets and liabilities, is a function of the asset allocation strategy and takes the form

$$E_t(\delta) = \sum_{k=1}^K \eta^k(\delta) S^k_t - L_t, \quad t \in [0,1].$$

As a consequence, both risk $\rho(E_1(\delta))$ and return $E_1(\delta)/E_0 - 1$ depend on the strategy δ.

3.1.2 Basis Asset-Liability Model

As a simple reference model we consider a Black-Scholes market and a single deterministic liability.

Asset Model. The financial market model consists of two liquidly tradable primary products: one riskless asset (*savings account*) and one risky asset (*stock*). Their price processes $(S^1_t)_{t\in[0,1]}$, $(S^2_t)_{t\in[0,1]}$ are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$ and follow the classical *Black-Scholes model*, i. e.,

- *Savings account*: $S_t^1 = \exp(rt)$, $t \in [0,1]$, with interest rate r,
- *Stock:* $S_t^2 = S_0^2 \exp(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t)$, $t \in [0,1]$, with $S_0^2 \in (0,\infty), \mu \in \mathbb{R}, \sigma \in (0,\infty)$,

where $(W_t)_{t\in[0,1]}$ denotes a one-dimensional Wiener process. Note that $\mathbb{E}[S_t^2] = S_0^2 \exp(\mu t)$. For the remaining part of the paper, we assume that the risk-free interest rate r equals zero, i. e., $S_t^1 = 1$, $t \in [0,1]$.

Liability Model. We assume that the insurance network sells a pure endowment with maturity 1 only. The network's premium income in $t = 0$ is denoted by π. The liabilities are deterministic, and the actuarial interest rate is assumed to be zero. Consequently, the actuarial reserve is a constant, i. e., $L_t = \pi$, $t \in [0,1]$.

In this basis setting, the net asset value is given by

$$E_t(\delta) = \eta^1(\delta)S_t^1 + \eta^2(\delta)S_t^2 - L_t = \eta^1(\delta) + \eta^2(\delta)S_t^2 - \pi \quad (t \in [0,1])$$

for any asset allocation $\delta \in \mathbb{R}^2, \delta^2 \geq 0, \delta^1 = 1-\delta^2 \geq 0$. Randomness is driven only by the terminal stock value S_1^2. This allows us to derive the minimal risk capital

$$\square_{i=1}^n \rho^i \left(E_1(\delta)\right)$$

for the three risk measures V@R, AV@R and RV@R in closed form.

Corollary 3.1. *Let $\rho^i = \mathrm{RV@R}_{\alpha_i,\beta_i}$, $\alpha_i, \beta_i \geq 0$, be the risk measure of network's entity i, $i = 1,\ldots,n$, and define $\alpha = \alpha_1 + \ldots + \alpha_n$, $\beta = \max\{\beta_1,\ldots,\beta_n\}$. Let $\delta \in \mathbb{R}^2$ be a fixed asset allocation strategy of the network. If $\alpha + \beta \leq 1$, then optimal risk sharing yields*

$$\begin{aligned}\square_{i=1}^n \mathrm{RV@R}_{\alpha_i,\beta_i}(E_1(\delta)) &= \mathrm{RV@R}_{\alpha,\beta}(E_1(\delta)) \\ &= -\eta^2(\delta)S_0^2 e^{\mu}\frac{1}{\beta}\left(\Phi(\Phi^{-1}(\alpha+\beta)-\sigma) - \Phi(\Phi^{-1}(\alpha)-\sigma)\right) - \eta^1(\delta) + \pi,\end{aligned}$$

where Φ denotes the cumulative distribution function of the standard normal distribution. In particular, the minimal SCRs take the form

$$\begin{aligned}\square_{i=1}^n \mathrm{SCR}^i_{\mathcal{A}}(E_1(\delta)) &= \eta^2(\delta)S_0^2\left(1 - e^{\mu}\frac{1}{\beta}\left(\Phi(\Phi^{-1}(\alpha+\beta)-\sigma) - \Phi(\Phi^{-1}(\alpha)-\sigma)\right)\right), \\ \square_{i=1}^n \mathrm{SCR}^i_{\mathrm{mean}}(E_1(\delta)) &= \eta^2(\delta)S_0^2\left(e^{\mu} - e^{\mu}\frac{1}{\beta}\left(\Phi(\Phi^{-1}(\alpha+\beta)-\sigma) - \Phi(\Phi^{-1}(\alpha)-\sigma)\right)\right).\end{aligned}$$

Proof. The proof is given in Section B. □

As a byproduct, Corollary 3.1 provides the corresponding results for V@R and AV@R.

Corollary 3.2. *Let $\delta \in \mathbb{R}^2$ be the network's asset allocation strategy.*

i) Let ρ^i be given by $\mathrm{V@R}_{\alpha_i}$, $\alpha_i \in (0,1)$, $i = 1,\ldots,n$, and set $\alpha = \alpha_1 + \ldots + \alpha_n$. If $\alpha \leq 1$, then

$$\square_{i=1}^n \mathrm{V@R}_{\alpha_i}(E_1(\delta)) = -\eta^2(\delta)S_0^2 e^{\mu}\exp\left(\Phi^{-1}(\alpha)\sigma - \frac{\sigma^2}{2}\right) - \eta^1(\delta) + \pi.$$

Moreover,

$$\begin{aligned}\square_{i=1}^n \mathrm{SCR}^i_{\mathcal{A}}(E_1(\delta)) &= \eta^2(\delta)S_0^2\left(1 - e^{\mu}\exp\left(\Phi^{-1}(\alpha)\sigma - \frac{\sigma^2}{2}\right)\right), \\ \square_{i=1}^n \mathrm{SCR}^i_{\mathrm{mean}}(E_1(\delta)) &= \eta^2(\delta)S_0^2\left(e^{\mu} - e^{\mu}\exp\left(\Phi^{-1}(\alpha)\sigma - \frac{\sigma^2}{2}\right)\right).\end{aligned}$$

ii) Let ρ^i be given by $\mathrm{AV@R}_{\beta_i}$, $\beta_i \in (0,1)$, $i = 1, \ldots, n$, *and define* $\beta = \max\{\beta_1, \ldots, \beta_n\}$. *If* $\beta \leq 1$, *then*

$$\Box_{i=1}^{n} \mathrm{AV@R}_{\beta_i}(E_1(\delta)) = -\eta^2(\delta) S_0^2 e^{\mu} \frac{1}{\beta} \Phi(\Phi^{-1}(\beta) - \sigma) - \eta^1(\delta) + \pi.$$

In particular, we have

$$\begin{aligned} \Box_{i=1}^{n} \mathrm{SCR}^i_{\mathcal{A}}(E_1(\delta)) &= \eta^2(\delta) S_0^2 \left(1 - e^{\mu} \frac{1}{\beta} \Phi(\Phi^{-1}(\beta) - \sigma)\right), \\ \Box_{i=1}^{n} \mathrm{SCR}^i_{\mathrm{mean}}(E_1(\delta)) &= \eta^2(\delta) S_0^2 \left(e^{\mu} - e^{\mu} \frac{1}{\beta} \Phi(\Phi^{-1}(\beta) - \sigma)\right). \end{aligned}$$

Proof. The proof is given in Section B. □

3.2 Parameterization

Let us now summarize our standing assumptions on the parameterization.

Remark 3.3. For V@R-type risk measures and for sufficiently many sub-entities, the network can reduce its total risk substantially, as described in Eq. (7). If the best case is unbounded, total risk will be equal to $-\infty$. Our case studies below rely on simulation methods with a finite number of samples. In all numerical experiments we run $500,000$ simulations. If the best case is unbounded, the sampled best case will always be a finite number, and it will thus not be possible to reproduce Eq. (7). For this reason, we modify all distributions in the extreme tails such that they will be of bounded support. This enables a simulation-based analysis of Eq. (7) and related results in our case studies. To be more precise, asset distributions are modified by setting asset values above the 99.95%-quantile to the 99.95%-quantile.

Analogously, we also modify liability distributions by setting liability values above the 99.95%-quantile to the 99.95%-quantile, and below the 0.05%-quantile to the 0.05%-quantile when extending the basis Asset-Liability model in Section 3.1.2 to cover also random liabilities and different dependence structures. This procedure will be applied to a liability distribution in Section 3.3.2 despite the fact that its support is bounded. This is done in order to avoid settings that require more sophisticated rare event simulation. The resulting, simplified model is well suited to numerically illustrate the effect of network size and ALM-strategies on risk. Alternative models in which extreme tail events possess a large influence on the outcome of simulations require more care in terms of simulation techniques for rare events. This is, however, not the focus of our paper. We thus concentrate on the described setting.

Details of the considered distributions are discussed below: The lognormal distribution refers to the asset side, i. e., a stock price, and is introduced in the next paragraph; the beta distribution captures random mortality, see Eq. (10); the stable distribution is introduced in Section 3.3.3 and models an additional asset with large downside risk. The relevant quantile values are provided in Table 2.

	Lognormal distr.	Beta distr.	Stable distr.
0.9995-quantile	66.2512	0.9718	3.2209
0.0005-quantile	not relevant	0.7775	not relevant

Table 2: Adjustment of distributions.

Parameterization of the Basis Model. For the asset side, we assume that the initial stock price is given by $S_0^2 = 30$ and that the stock price dynamics is determined by the drift $\mu = \ln(35/30) \approx 0.1542$ (i. e., $\mathbb{E}[S_1^2] = 35$) and the volatility $\sigma = 0.2$. As discussed in Remark 3.3, we bound the asset value by its 99.95%-quantile which equals 66.2512, see Table 2, by modifying its distribution as explained. Interest rates of the savings account are assumed to be zero. On the liability side, we assume that the network's premium income in $t = 0$ is $\pi = 90$. Since the liabilities are deterministic, this implies $L_0 = L_1 = \pi = 90$.

The initial equity value is set to $E_0 = 30$. In this case, the total asset amount of the balance sheet is given by $E_0 + L_0 = 120$. The network's asset allocation $\delta \in \mathbb{R}^2$ is assumed to be fixed and set to $\delta^1 = 0.75$ and $\delta^2 = 0.25$, i. e., we obtain the corresponding numbers of shares

$$\eta^1(\delta) = 90, \quad \eta^2(\delta) = 1.$$

Note that this asset allocation yields the terminal equity

$$E_1(\delta) = \eta^1(\delta) + \eta^2(\delta) S_1^2 - \pi = S_1^2$$

proportional to the stock value. In particular, for the given positive drift μ, we have

$$\mathbb{E}[E_1(\delta)] = \mathbb{E}[S_1^2] = S_0^2 \exp(\mu) > S_0^2 = E_0,$$

i. e.,

$$\mathrm{SCR}_{\mathcal{A}}(E_1(\delta)) = E_0 + \rho(E_1(\delta)) < \mathbb{E}[E_1(\delta)] + \rho(E_1(\delta)) = \mathrm{SCR}_{\mathrm{mean}}(E_1(\delta))$$

for any monetary risk measure ρ.

Parameterization of Risk Measures. Our case studies compare and analyze the effect of optimal risk sharing for three different risk measures: V@R, AV@R and RV@R. We assume that within the network all firms use the same risk measure with the same parameters, i. e.,

(a) $\rho^i = \mathrm{V@R}_\alpha$, $\alpha \in (0,1)$, for all $i = 1, \ldots, n$,

(b) $\rho^i = \mathrm{AV@R}_\beta$, $\beta \in (0,1)$, for all $i = 1, \ldots, n$,

(c) $\rho^i = \mathrm{RV@R}_{\gamma,\epsilon}$, $\gamma, \epsilon \in (0,1)$, for all $i = 1, \ldots, n$.

This situation might result from a management decision to apply a unified risk measurement approach within the network, or it could be enforced by regulatory requirements if all firms are subject to the same regulation scheme.

For value at risk, we choose the level $\alpha = 0.1$, and we fix $\gamma = 0.05$ for the range value at risk. To ensure comparability of results between the three risk measures, the remaining parameters β, ϵ are calibrated such that for $X \sim \mathcal{N}(0,1)$ with cumulative distribution function Φ and probability density function φ

$$\mathrm{V@R}_\alpha(X) = \mathrm{AV@R}_\beta(X) = \mathrm{RV@R}_{\gamma,\epsilon}(X). \tag{8}$$

For this purpose, we use that $\mathrm{V@R}_\alpha(X) = -q_X^+(\alpha) = -\Phi^{-1}(\alpha)$,

$$\begin{aligned} \mathrm{AV@R}_\beta(X) &= \tfrac{1}{\beta} \int_0^\beta \mathrm{V@R}_\alpha(X)\, d\alpha = -\tfrac{1}{\beta} \int_0^\beta \Phi^{-1}(\alpha)\, d\alpha \\ &= -\tfrac{1}{\beta} \int_{-\infty}^{\Phi^{-1}(\beta)} y\varphi(y)\, dy = -\tfrac{1}{\beta} \varphi(\Phi^{-1}(\beta)), \end{aligned}$$

due to the substitution $y = \Phi^{-1}(\alpha)$ and $\varphi'(y) = y\varphi(y)$, and

$$\mathrm{RV@R}_{\gamma,\epsilon}(X) = \tfrac{1}{\epsilon} \int_\gamma^{\gamma+\epsilon} \mathrm{V@R}_\alpha(X)\, d\alpha = -\tfrac{1}{\epsilon} \int_\gamma^{\gamma+\epsilon} \Phi^{-1}(\alpha)\, d\alpha = -\tfrac{1}{\epsilon} \left(\varphi(\Phi^{-1}(\gamma+\epsilon)) - \varphi(\Phi^{-1}(\gamma)) \right).$$

Solving Eq. (8) with these formulae for given α and γ numerically yields the following parameters:

$\mathrm{V@R}_\alpha$	$\mathrm{AV@R}_\beta$	$\mathrm{RV@R}_{\gamma,\epsilon}$
$\alpha = 0.1$	$\beta = 0.2456$	$\gamma = 0.05, \epsilon = 0.1072$

Table 3: Parameterization of risk measures.

Remark 3.4. Observe that the chosen quantile levels in Remark 3.3 and Table 2 are in the extreme tail of the distributions, if compared to the parameters of the risk measures in Table 3. Consequently, also within our modified model with adjusted distributions the chosen risk measures are non-trivial functionals of the tails.

3.3 Numerical Case Studies

3.3.1 Unsophisticated Network vs. Sophisticated Network

Let us first consider the basis ALM model with deterministic liabilities. The first row in Tables 4–6 displays the risk capital $\rho(E_1(\delta))$ and the corresponding SCRs for V@R, AV@R and RV@R of a single firm. This corresponds to the consolidated case and can be interpreted as an unsophisticated network. All values are almost equal across different risk measures due to the applied standardization of the risk levels $\alpha, \beta, \gamma, \epsilon$, although $E_1(\delta)$ follows a lognormal distribution instead of a standard normal distribution.

Sophisticated networks may, firstly, adjust their structure by increasing the number of entities n. For a fixed number n of firms, the corporate network will, secondly, design optimal intra-network capital transfers that minimize the total risk in Eq. (1). The second and the third row in Tables 4–6 quantify the effect on risk capital and on the corresponding SCRs for $n = 5$ and $n = 10$ firms:

- For the two risk measures of V@R-type, V@R and RV@R, we observe that downside risk can be reduced significantly by optimal capital transfers that hide the tail risk. For n sufficiently large, the corporate network could even determine a capital allocation such that

$$\square_{i=1}^{n} \rho^i(E_1(\delta)) = -\operatorname{ess\,sup} E_1(\delta),$$

  corresponding to the best case scenario. This requires $n \cdot \alpha \geq 1$ for the risk measure $\mathrm{V@R}_\alpha$ and $n \cdot \gamma \geq 1$ for $\mathrm{RV@R}_{\gamma,\epsilon}$ (cf. page 6). For $\mathrm{V@R}_\alpha$ with $\alpha = 0.1$, this condition is already satisfied for a number of firms $n \geq 10$, and the simulations provide the expected result.

- In contrast, for the coherent risk measure AV@R, optimal risk sharing does, of course, not reduce the risk capital – hiding tail risk is not possible.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^{n}\mathrm{V@R}_\alpha^i\,(E_1(\delta))$	$\square_{i=1}^{n}\mathrm{SCR}_\mathcal{A}^i\,(E_1(\delta))$	$\square_{i=1}^{n}\mathrm{SCR}_{\mathrm{mean}}^i\,(E_1(\delta))$
$n=1$	34.9982	-26.5577	3.4423	8.4405
$n=5$	34.9982	-34.3060	-4.3060	0.6922
$n=10$	34.9982	-66.2512	-36.2512	-31.2530

Table 4: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\mathrm{V@R}_{0.1}$; deterministic liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^{n}\mathrm{AV@R}_\beta^i\,(E_1(\delta))$	$\square_{i=1}^{n}\mathrm{SCR}_\mathcal{A}^i\,(E_1(\delta))$	$\square_{i=1}^{n}\mathrm{SCR}_{\mathrm{mean}}^i\,(E_1(\delta))$
$n=1,5,10$	34.9982	-26.6784	3.3216	8.3198

Table 5: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\mathrm{AV@R}_{0.2456}$; deterministic liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^{n}\mathrm{RV@R}_{\gamma,\epsilon}^i\,(E_1(\delta))$	$\square_{i=1}^{n}\mathrm{SCR}_\mathcal{A}^i\,(E_1(\delta))$	$\square_{i=1}^{n}\mathrm{SCR}_{\mathrm{mean}}^i\,(E_1(\delta))$
$n=1$	34.9982	-26.5722	3.4278	8.4260
$n=5$	34.9982	-30.9523	-0.9523	4.0459
$n=10$	34.9982	-35.2473	-5.2473	-0.2491

Table 6: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\mathrm{RV@R}_{0.05,0.1072}$; deterministic liabilities.

3.3.2 Random Liabilities

We extend the basis ALM model by including random liabilities. The insurance network is assumed to sell pure endowment contracts only, i. e., a product depending on the random future life time of the insurees. The idiosyncratic risk of individuals becomes irrelevant in a very large pool, but the systematic risk, random mortality, does not average out. This is the focus of the case study.

For insured persons aged x, we denote by p_x^* and p_x their one-year *actuarial* survival probability and their one-year *random* survival probability, respectively. We use the assumption that the actuarial survival probability p_x^* is the *best estimate* of the random survival probability in the sense that $\mathbb{E}[p_x] = p_x^*$ and that p_x^* does not yet include any margin for unexpected losses, i. e., deviations from the expected value. In this case, for a sum insured $L > 0$, the premium is calculated as $\pi = L \cdot p_x^*$, and the random liabilities at time $t = 1$ are given by

$$L_1 = L \cdot p_x = \frac{p_x}{p_x^*} \pi.$$

The last term corresponds to the actuarial reserve adapted to mortality by an appropriate multiplier, the ratio of random and actuarial survival probability.

For zero interest rates, the network's random equity at time $t = 1$ is then given by

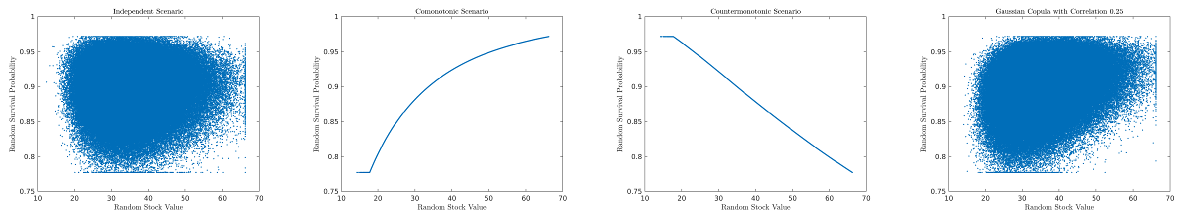
$$E_1(\delta) = \eta^1(\delta) + \eta^2(\delta) S_1^2 - L_1 = \eta^1(\delta) + \eta^2(\delta) S_1^2 - \frac{p_x}{p_x^*} \pi. \tag{9}$$

The extended model (9) reduces to the basis ALM model if $p_x \equiv p_x^*$ is deterministic. For a monetary risk measure ρ, the risk

$$\rho\left(E_1(\delta)\right) = \rho\left(\eta^2(\delta) S_1^2 - \frac{p_x}{p_x^*} \pi\right) - \eta^1(\delta)$$

accounts for both the network's asset risk and biometric risk, i. e., the longevity of policyholders.

We analyze the network's optimal risk sharing strategy for four different dependence structures of assets (stock) and liabilities: *independence*, *comonotonicity*, *countermonotonicity*, and a dependency modeled by a *Gaussian copula* with correlation parameter 0.25. These dependencies are illustrated in Figure 1. We do not claim that pure comonotonicity and countermonotonicity are realistic, but study them to illustrate the implications of particularly extreme forms of dependence. The case of the Gaussian copula corresponds to the specifications of the Solvency II Standard Formula which prescribes - implicitly embedded into a multivariate Gaussian setting - a linear correlation of 0.25 between market risk and underwriting risk life for risk aggregation by square-root-formula, cf. European Commission (2009), Annex IV.



(a) (b) (c) (d)

Figure 1: Dependence structures: The stock value and the survival probability are (a) independent, (b) comonotonic, (c) countermonotonic, and (d) follow a Gaussian copula with correlation 0.25.

Independent assets and liabilities do not affect each other. In the comonotonic case, asset and liability values change in the same direction. In particular, increasing liabilities are associated with increasing asset values such that increasing costs for the insurer are hedged by gains in the financial market. In contrast, countermonotonic assets and liabilities correspond to a scenario in which increasing liability values are associated with decreasing asset values. The first situation could, for example, correspond to a scenario of joint technical and medical innovation with both increased wealth and longevity. The second situation could be associated with medical innovation and longevity coupled with an aging population that liquidates assets to generate liquidity.

Countermonotonic assets and liabilities are problematic, since high insurance claims occur together with low asset values and yield a low book value of equity of insurers. In the worst case, the liabilities might not be covered by the asset value anymore. The Gaussian copula with correlation 0.25 basically corresponds to a minor positive linear dependency between the asset and liability side. In this respect, this case interpolates between independence and comonotonicity.

For the numerical results, we rely on the parameterization of Section 3.2. In addition, we assume a sum insured $L = 100$, $p_x^* = 0.9$ and

$$p_x \sim \text{Beta}(90, 10). \tag{10}$$

Then, $\mathbb{E}[p_x] = p_x^* = 0.9$, and hence $\mathbb{E}[L_1] = \pi = L_0$. As described in Remark 3.3, we modify this distribution in the tails. The relevant quantiles are shown in Table 2. For the sake of comparison to the basis case in Section 3.3.1, we calibrate the asset allocation δ^1, δ^2 with $\delta^1 + \delta^2 = 1$ such that for a network with a single firm only and for independent assets and liabilities $\text{V@R}_\alpha(E_1(\delta))$ coincides with the basis ALM model. This yields

$$\delta^1 = 0.8382, \delta^2 = 0.1618,$$

i. e., the fraction δ^1 in the savings account is now higher. This is not surprising since random mortality increases risk which needs to be offset by a reduction of the stock investment. As a consequence, the return decreases as well; the expected value of the future net asset value is:

$$\mathbb{E}[E_1(\delta)] = (E_0 + L_0)\left(\delta^1 \mathbb{E}\left[\frac{S_1^1}{S_0^1}\right] + \delta^2 \mathbb{E}\left[\frac{S_1^2}{S_0^2}\right]\right) - \mathbb{E}[L_1] = (E_0 + L_0)\left(\delta^1 + \delta^2 \exp(\mu)\right) - \pi.$$

The following tables summarize the numerical results.

Case Study I - Independent Stock and Liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \text{V@R}_\alpha^i\,(E_1(\delta))$	$\square_{i=1}^n \text{SCR}_{\mathcal{A}}^i\,(E_1(\delta))$	$\square_{i=1}^n \text{SCR}_{\text{mean}}^i\,(E_1(\delta))$
$n = 1$	33.2400	-26.5578	3.4422	6.6822
$n = 5$	33.2400	-32.8451	-2.8451	0.3949
$n = 10$	33.2400	-65.7126	-35.7126	-32.4726

Table 7: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\text{V@R}_{0.1}$; independent assets and liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \text{AV@R}_\beta^i\,(E_1(\delta))$	$\square_{i=1}^n \text{SCR}_{\mathcal{A}}^i\,(E_1(\delta))$	$\square_{i=1}^n \text{SCR}_{\text{mean}}^i\,(E_1(\delta))$
$n = 1, 5, 10$	33.2400	-26.6353	3.3647	6.6047

Table 8: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\text{AV@R}_{0.2456}$; independent assets and liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \text{RV@R}_{\gamma,\epsilon}^i\,(E_1(\delta))$	$\square_{i=1}^n \text{SCR}_{\mathcal{A}}^i\,(E_1(\delta))$	$\square_{i=1}^n \text{SCR}_{\text{mean}}^i\,(E_1(\delta))$
$n = 1$	33.2400	-26.5684	3.4316	6.6715
$n = 5$	33.2400	-30.1805	-0.1805	3.0595
$n = 10$	33.2400	-33.5769	-3.5769	-0.3370

Table 9: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\text{RV@R}_{0.05,0.1072}$; independent assets and liabilities.

Case Study II - Comonotonic Stock and Liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \mathrm{V@R}^i_\alpha\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}^i_\mathcal{A}\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}^i_{\mathrm{mean}}\,(E_1(\delta))$
$n=1$	33.2343	-31.7546	-1.7546	1.4797
$n=5$	33.2343	-32.5588	-2.5588	0.6755
$n=10$	33.2343	-46.2791	-16.2791	-13.0448

Table 10: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\mathrm{V@R}_{0.1}$; comonotonic assets and liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \mathrm{AV@R}^i_\beta\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}^i_\mathcal{A}\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}^i_{\mathrm{mean}}\,(E_1(\delta))$
$n=1,5,10$	33.2343	-31.7879	-1.7879	1.4464

Table 11: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\mathrm{AV@R}_{0.2456}$; comonotonic assets and liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \mathrm{RV@R}^i_{\gamma,\epsilon}\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}^i_\mathcal{A}\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}^i_{\mathrm{mean}}\,(E_1(\delta))$
$n=1$	33.2343	-31.7601	-1.7601	1.4742
$n=5$	33.2343	-32.0290	-2.0290	1.2053
$n=10$	33.2343	-32.7668	-2.7668	0.4675

Table 12: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\mathrm{RV@R}_{0.05,0.1072}$; comonotonic assets and liabilities.

Case Study III - Countermonotonic Stock and Liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \mathrm{V@R}^i_\alpha\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}^i_\mathcal{A}\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}^i_{\mathrm{mean}}\,(E_1(\delta))$
$n=1$	33.2365	-24.1537	5.8463	9.0828
$n=5$	33.2365	-32.5189	-2.5189	0.7177
$n=10$	33.2365	-65.7126	-35.7126	-32.4760

Table 13: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\mathrm{V@R}_{0.1}$; countermonotonic assets and liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \mathrm{AV@R}^i_\beta\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}^i_\mathcal{A}\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}^i_{\mathrm{mean}}\,(E_1(\delta))$
$n=1,5,10$	33.2365	-24.3001	5.6999	8.9365

Table 14: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\mathrm{AV@R}_{0.2456}$; countermonotonic assets and liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \mathrm{RV@R}^i_{\gamma,\epsilon}\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}^i_\mathcal{A}\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}^i_{\mathrm{mean}}\,(E_1(\delta))$
$n=1$	33.2365	-24.1789	5.8211	9.0577
$n=5$	33.2365	-28.8983	1.1017	4.3382
$n=10$	33.2365	-33.5348	-3.5348	-0.2982

Table 15: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\mathrm{RV@R}_{0.05,0.1072}$; countermonotonic assets and liabilities.

Case Study IV - Gaussian Copula with Correlation 0.25.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \mathrm{V@R}_\alpha^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_\mathcal{A}^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_{\mathrm{mean}}^i\,(E_1(\delta))$
$n=1$	33.2289	-27.3255	2.6745	5.9034
$n=5$	33.2289	-32.9015	-2.9015	0.3274
$n=10$	33.2289	-64.0618	-34.0618	-30.8328

Table 16: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\mathrm{V@R}_{0.1}$; Gaussian copula with correlation 0.25.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \mathrm{AV@R}_\beta^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_\mathcal{A}^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_{\mathrm{mean}}^i\,(E_1(\delta))$
$n=1,5,10$	33.2289	-27.3935	2.6065	5.8355

Table 17: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\mathrm{AV@R}_{0.2456}$; Gaussian copula with correlation 0.25.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \mathrm{RV@R}_{\gamma,\epsilon}^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_\mathcal{A}^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_{\mathrm{mean}}^i\,(E_1(\delta))$
$n=1$	33.2289	-27.3377	2.6623	5.8912
$n=5$	33.2289	-30.5547	-0.5547	2.6743
$n=10$	33.2289	-33.5492	-3.5492	-0.3203

Table 18: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\mathrm{RV@R}_{0.05,0.1072}$; Gaussian copula with correlation 0.25.

In the consolidated case – corresponding to an unsophisticated network consisting of a single firm only – and for all three risk measures V@R, AV@R and RV@R, the associated risk capital $\rho(E_1(\delta))$ reflects the different dependence structures in the following sense: The highest risk capital is attained for the countermonotonic case, the lowest risk capital is observed for the comonotonic case, while the risk capital for independent assets and liabilities is between the values of the two extreme dependency structures. A Gaussian copula with correlation 0.25 – corresponding to the standard model in Solvency II – is very close to the independent case.

In analogy to Section 3.3.1, the numerical results illustrate for all four dependence structures that optimal capital transfers within a sophisticated network hide the downside risk, if capital regulation is based on V@R-type risk measures such as V@R and RV@R. In contrast, there is no reduction of risk capital by optimal risk sharing for the coherent risk measure AV@R.

For V@R-type risk measures, the different levels of risk capital for the countermonotonic and independent case disappear for increasing n. The difference of the inf-convolutions $\square_{i=1}^n \mathrm{V@R}_\alpha^i\,(E_1(\delta))$ in the countermonotonic and the independent case decreases from 2.4 for $n = 1$ to 0.3 for $n = 5$ and finally to 0 for $n = 10$. Similarly, the difference of the inf-convolutions $\square_{i=1}^n \mathrm{RV@R}_{\gamma,\epsilon}^i\,(E_1(\delta))$ in the countermonotonic and the independent case decreases from 2.4 for $n = 1$ to 1.3 for $n = 5$ and finally to nearly 0 for $n = 10$. Observe that $\square_{i=1}^{10} \mathrm{V@R}_\alpha^i\,(E_1(\delta))$ equals -65.71 for both the countermonotonic and the independent case, corresponding to the best case – as known from Eq. (7).

3.3.3 Left-Tailed Asset

In this section, we consider again the basis ALM model with deterministic liabilities as described in Section 3.1, but extend the financial market by including a third *left-tailed*, also called *left-skewed*, asset with price process $(S_t^3)_{t\in[0,1]}$. This asset is characterized by a skewed distribution with the possibility of losses and – in comparison to the stock – a higher downside risk. More precisely, its price process is modeled by

$$S_t^3 = S_0^3 \exp(\zeta t) + Z - \mathbb{E}[Z], \quad t \in (0, 1],$$

where the initial value $S_0^3 > 0$ is a fixed constant, $\zeta > 0$ is a rate of exponential growth, and Z is a random variable with stable distribution.

Definition 3.5. A random variable Z has a *stable distribution* $\mathcal{S}(a,b,c,d)$ with parameters $a \in (0,2], b \in [-1,1], c \in (0,\infty), d \in \mathbb{R}$, i. e., $Z \sim \mathcal{S}(a,b,c,d)$, if its characteristic function is given by

$$\mathbb{E}\left[e^{isZ}\right] = \begin{cases} \exp\left(-c^\alpha |s|^a \left[1 + ib\,\mathrm{sign}(s) \tan\frac{\pi a}{2}\left(\left(c|s|^{1-a} - 1\right)\right)\right] + ids\right), & a \neq 1, \\ \exp\left(-c|s|\left[1 + ib\,\mathrm{sign}(s) \tan\frac{2}{\pi}\left(c|s|\right)\right] + ids\right), & a = 1. \end{cases}$$

For the numerical case study, we fix $S_0^3 = 1$, $\zeta = 0.3$ and assume that

$$Z \sim \mathcal{S}(1.5, -1, 1, 0)$$

is independent from the stock price process $(S_t^2)_{t\in[0,1]}$. Figure 2 shows the probability density function of Z. Again, as explained in Remark 3.3, we modify the distribution such that it is bounded from above. Table 2 shows the new upper bound of 3.22 at the 99.95%-quantile of the original distribution. Note that $\mathbb{E}[S_1^3/S_0^3] \approx \exp(\zeta) > \exp(\mu) \approx \mathbb{E}[S_1^2/S_0^2]$ for the parameters $\zeta = 0.3$ and $\mu = 0.1542$, i. e., the expected return of the left-tailed asset exceeds the expected return of the stock, compensating for the higher risk of this position.

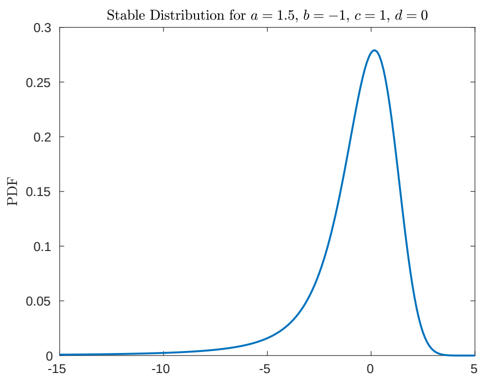


Figure 2: PDF of $Z \sim \mathcal{S}(1.5, -1, 1, 0)$.

For three assets, the book value of equity at terminal time 1 is given by

$$\begin{aligned} E_1(\delta) &= \eta^1(\delta) + \eta^2(\delta) S_1^2 + \eta^3(\delta) S_1^3 - \pi \\ &= (E_0 + L_0)\left(\delta^1 \frac{S_1^1}{S_0^1} + \delta^2 \frac{S_1^2}{S_0^2} + \delta^3 \frac{S_1^3}{S_0^3}\right) - \pi, \quad \delta \in \mathbb{R}^3, \delta^1, \delta^2, \delta^3 \geq 0, \delta^1 + \delta^2 + \delta^3 = 1, \end{aligned}$$

where $\eta^1(\delta)$ and $\eta^2(\delta)$ are as before and $\eta^3(\delta)$ denotes the number of left-tailed assets bought at time $t = 0$. Thus, a higher fraction δ^3 yields a higher expected terminal net asset value $\mathbb{E}[E_1(\delta)]$, but is associated with a higher downside risk.

Case Study I - Fixed Asset Allocation Including a Left-Tailed Asset. Let us first analyze the impact of the left-tailed asset on network risk minimization for a fixed asset allocation $\delta \in \mathbb{R}^3$, where a small fraction $\delta^3 = 0.01$ is invested in the left-tailed asset. For the sake of comparison to the basis case in Section 3.3.1, we calibrate the remaining fractions δ^1, δ^2 with $\delta^1 + \delta^2 + \delta^3 = 1$ such that for the consolidated case, i. e., a network with only a single firm, $\mathrm{V@R}_\alpha(E_1(\delta))$ coincides with the basis case. This yields the allocation

$$\delta^1 = 0.73901, \delta^2 = 0.2510, \delta^3 = 0.01.$$

In analogy to Section 3.3.1, the numerical results in Tables 19, 20 & 21 illustrate that optimal capital transfers within a sophisticated network hide the downside risk, if capital regulation is based on V@R-type risk measures such as V@R and RV@R.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \text{V@R}_\alpha^i\,(E_1(\delta))$	$\square_{i=1}^n \text{SCR}_{\mathcal{A}}^i\,(E_1(\delta))$	$\square_{i=1}^n \text{SCR}_{\text{mean}}^i\,(E_1(\delta))$
$n=1$	35.4378	-26.5577	3.4423	8.8801
$n=5$	35.4378	-35.1833	-5.1833	0.2545
$n=10$	35.4378	-71.8246	-41.8246	-36.3867

Table 19: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\text{V@R}_{0.1}$; additional left-tailed asset.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \text{AV@R}_\beta^i\,(E_1(\delta))$	$\square_{i=1}^n \text{SCR}_{\mathcal{A}}^i\,(E_1(\delta))$	$\square_{i=1}^n \text{SCR}_{\text{mean}}^i\,(E_1(\delta))$
$n=1,5,10$	35.4378	-25.4473	4.5527	9.9905

Table 20: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\text{AV@R}_{0.2456}$; additional left-tailed asset.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \text{RV@R}_{\gamma,\epsilon}^i\,(E_1(\delta))$	$\square_{i=1}^n \text{SCR}_{\mathcal{A}}^i\,(E_1(\delta))$	$\square_{i=1}^n \text{SCR}_{\text{mean}}^i\,(E_1(\delta))$
$n=1$	35.4378	-26.5512	3.4488	8.8866
$n=5$	35.4378	-31.5879	-1.5879	3.8499
$n=10$	35.4378	-36.1717	-6.1717	-0.7339

Table 21: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\text{RV@R}_{0.05,0.1072}$; additional left-tailed asset.

Case Study II - Optimizing the Asset Allocation. In the second step, we fix the fraction $\delta^1 = 0.75$ invested in the savings account and vary the fraction δ^3 held in the left-tailed asset (and $\delta^2 = 1 - \delta^1 - \delta^3$, respectively) in the range $[0, 0.25]$. The left boundary point $\delta^3 = 0$ corresponds to the basis ALM model in Section 3.3.1, i. e., there is no investment in the left-tailed asset and the full remaining fraction $\delta^2 = 0.25$ of asset amount of the balance sheet is invested in the stock. As an anchor point, the first row in Tables 22–24 coincides with the numerical results of the basis ALM model (cf. Tables 4–6). For the right boundary point $\delta^3 = 0.25$, the fraction of 0.25 invested initially in the stock is completely replaced by the left-tailed asset. For different risk measures and different numbers of entities we vary the fraction δ^3 held in the left-tailed asset. For each of these cases, an approximately optimal value δ^3 that minimizes the total risk measurement within the considered range can be determined from the tables. Risk sharing is again optimally designed according to Theorem 2.4 in Weber (2018). cf. page 6.

Table 22 displays the expected terminal net asset value and the risk capital for the consolidated case, i. e., an unsophisticated network, for varying δ^3. A higher fraction δ^3 increases the expected terminal equity $\mathbb{E}[E_1(\delta)]$, thus the expected profit of the network. At the same time, substituting the stock by the left-tailed asset substantially increases risk capital for all three risk measures V@R, AV@R and RV@R. The risk measure AV@R is most sensitive to the re-allocation between stock and left-tailed asset.

Tables 23 & 24 show the relevant quantities for a sophisticated network which splits into $n = 5$ or $n = 10$ entities. Returns increase with δ^3, i. e., the fraction in the left-tailed asset, and are independent of n. However, with increasing n for the two V@R-type risk measures, V@R and RV@R, required capital decreases substantially. The effect of reduction is stronger for $\delta^3 > 0$ than in the basis ALM model corresponding to $\delta^3 = 0$. For $\delta^3 = 0$, the difference in $\square_{i=1}^n \text{V@R}_\alpha^i\,(E_1(\delta))$ for $n = 1$ and $n = 10$ is equal to 39.6, but for $\delta^3 = 0.25$ equal to 190.9. In the case of RV@R the corresponding differences are smaller but qualitatively similar, i. e., 8.7 and 93.8, respectively. In particular, the increase of risk capital for a single firm driven by investments in the left-tailed asset can completely be compensated within a sophisticated network.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \mathrm{V@R}_\alpha^i(E_1(\delta))$	$\square_{i=1}^n \mathrm{AV@R}_\beta^i(E_1(\delta))$	$\square_{i=1}^n \mathrm{RV@R}_{\gamma,\epsilon}^i(E_1(\delta))$
$\delta^3 = 0$	34.9982	-26.6784	-26.6822	-26.5722
$\delta^3 = 0.05$	36.0977	-21.6074	-12.7634	-21.2202
$\delta^3 = 0.1$	37.1972	-10.8346	7.1190	-10.0190
$\delta^3 = 0.15$	38.2967	0.7034	27.6693	1.9167
$\delta^3 = 0.2$	39.3962	12.4657	48.4084	14.0584
$\delta^3 = 0.25$	40.4958	24.2745	69.2267	26.2904

Table 22: Expected equity and minimized network risk capital for V@R, AV@R and RV@R for a split into $n = 1$ firms; additional left-tailed asset.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \mathrm{V@R}_\alpha^i(E_1(\delta))$	$\square_{i=1}^n \mathrm{AV@R}_\beta^i(E_1(\delta))$	$\square_{i=1}^n \mathrm{RV@R}_{\gamma,\epsilon}^i(E_1(\delta))$
$\delta^3 = 0$	34.9982	-34.3060	-26.6784	-30.9523
$\delta^3 = 0.05$	36.0977	-39.6707	-12.7634	-33.7124
$\delta^3 = 0.1$	37.1972	-45.4260	7.1190	-34.6340
$\delta^3 = 0.15$	38.2967	-50.9053	27.6693	-34.9642
$\delta^3 = 0.2$	39.3962	-56.2971	48.4084	-35.1112
$\delta^3 = 0.25$	40.4958	-61.6229	69.2267	-35.1748

Table 23: Expected equity and minimized network risk capital for V@R, AV@R and RV@R for a split into $n = 5$ firms; additional left-tailed asset.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \mathrm{V@R}_\alpha^i(E_1(\delta))$	$\square_{i=1}^n \mathrm{AV@R}_\beta^i(E_1(\delta))$	$\square_{i=1}^n \mathrm{RV@R}_{\gamma,\epsilon}^i(E_1(\delta))$
$\delta^3 = 0$	34.9982	-66.2512	-26.6784	-35.2473
$\delta^3 = 0.05$	36.0977	-86.1500	-12.7634	-41.0915
$\delta^3 = 0.1$	37.1972	-106.0488	7.1190	-47.8593
$\delta^3 = 0.15$	38.2967	-125.9476	27.6693	-54.4418
$\delta^3 = 0.2$	39.3962	-145.8646	48.4084	-60.9784
$\delta^3 = 0.25$	40.4958	-166.5765	69.2267	-67.5032

Table 24: Expected equity and minimized network risk capital for V@R, AV@R and RV@R for a split into $n = 10$ firms; additional left-tailed asset.

But the numerical results are even more striking. For $n = 1$ all risk measures indicate that investments into the left-tailed asset increase risk. The coherent risk measure AV@R is invariant under an increase of the number of entities. But for $n = 5$ and $n = 10$ both V@R-type risk measures lead to decreasing measurements of total risk if the fraction δ^3 invested in the left-tailed asset is increased. In the case of V@R, total risk $\square_{i=1}^n \mathrm{V@R}_\alpha^i(E_1(\delta))$ increases for $n = 1$ from -26.7 for $\delta^3 = 0$ to 24.3 for $\delta^3 = 0.25$, but decreases for $n = 5$ from -34.3 for $\delta^3 = 0$ to -61.6 for $\delta^3 = 0.25$ and for $n = 10$ from -66.3 for $\delta^3 = 0$ to -166.6 for $\delta^3 = 0.25$. A similar phenomenon is observed for RV@R, but less significant. Total risk $\square_{i=1}^n \mathrm{RV@R}_{\gamma,\epsilon}^i(E_1(\delta))$ increases for $n = 1$ from -26.6 for $\delta^3 = 0$ to 26.3 for $\delta^3 = 0.25$, but decreases for $n = 5$ from -31.0 for $\delta^3 = 0$ to -35.2 for $\delta^3 = 0.25$ and for $n = 10$ from -35.2 for $\delta^3 = 0$ to -67.5 for $\delta^3 = 0.25$. In particular, investing the maximum amount $\delta^3 = 0.25$ into the left-tailed asset yields the highest expected equity $\mathbb{E}[E_1(\delta)] = 40.4958$, while optimal risk sharing between $n = 5, 10$ entities yields the lowest risk capital with respect to V@R and RV@R.

Figure 3 illustrates the impact of the fraction δ^3 invested in the left-tailed asset on $\mathbb{E}[E_1(\delta)]$, $\square_{i=1}^n \rho^i(E_1(\delta))$, $\square_{i=1}^n \mathrm{SCR}_{\mathcal{A}}^i(E_1(\delta))$ and $\square_{i=1}^n \mathrm{SCR}_{\mathrm{mean}}^i(E_1(\delta))$ for $n = 1, 5, 10$ firms.

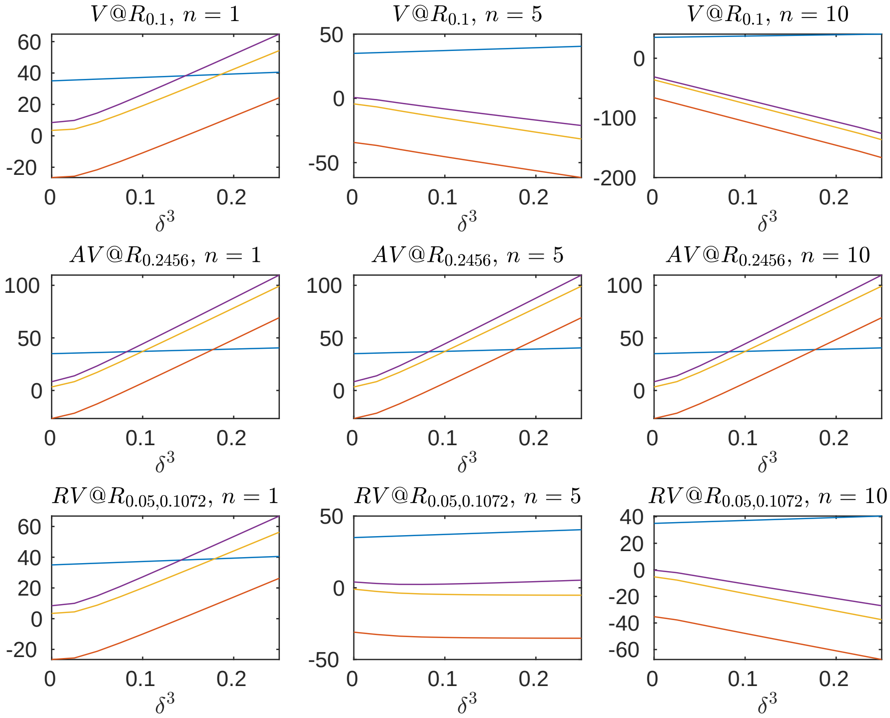


Figure 3: $\mathbb{E}[E_1(\delta)]$ in blue, $\square_{i=1}^n \rho^i\left(E_1(\delta)\right)$ in red, $\square_{i=1}^n \mathrm{SCR}^i_{\mathcal{A}}\left(E_1(\delta)\right)$ in yellow and $\square_{i=1}^n \mathrm{SCR}^i_{\mathrm{mean}}\left(E_1(\delta)\right)$ in purple.

Optimal risk sharing for V@R-type risk measures suggests that the network's management should invest as much as possible in the left-tailed asset and provides incentives for highly risky investments, i. e. – from a regulatory point of view – for risk mismanagement. In fact, the left-tailed asset is associated with a significant downside risk, as indicated by the coherent risk measure AV@R. In contrast to V@R-type risk measures, an asset allocation decision based on this coherent risk measure would avoid too large investments in the left-tailed asset.

4 Conclusion

Unless a consolidated solvency balance sheet is required, corporate networks may largely hide their total risk, if downside risk is quantified by risk measures of V@R-type – which includes the industry's standard risk measure value at risk. More precisely, a corporate network consisting of sufficiently many firms can largely reduce its total solvency capital requirement via optimal intra-network capital transfers and asset-liability management strategies. The size of capital reduction is increasing in the number n of firms in the network. If n is sufficiently large, the network can design a capital allocation such that the optimal network risk $\square_{i=1}^n \rho^i(E_1)$ coincides with $-\operatorname{ess\,sup} E_1$, corresponding to the best case scenario.

This paper illustrates the impact of optimal intra-network capital transfers embedded into a general asset-liability management model, allowing for different asset allocation strategies, random liabilities with different dependencies between assets and liabilities, and investments in a left-tailed asset. The numerical case studies show that V@R-type risk measures provide incentives for risky investments. In contrast, if risk management is based on the coherent risk measure average value at risk, downside risk cannot be hidden and misleading incentives are not present.

References

Acciaio, B. (2005), Two Problems Related to Utility Theory Under Unusual Assumptions, PhD thesis, University of Perugia.

Acciaio, B. (2007), 'Optimal Risk Sharing with Non-Monotone Monetary Functions', *Finance & Stochastics* **11**(2), 267–289.

Artzner, P., F. Delbaen, J.-M. Eber & D. Heath (1999), 'Coherent Measures of Risk', *Mathematical Finance* **9**(3), 203–228.

Asimit, A. V., A. M. Badescu & A. Tsanakas (2013), 'Optimal Risk Transfers in Insurance Groups', *European Actuarial Journal* **3**(1), 159–190.

Barrieu, P. & N. El Karoui (2005), 'Inf-Convolution of Risk Measures and Optimal Risk Transfer', *Finance and Stochastics* **9**, 269–298.

Barrieu, P. & N. El Karoui (2008), Pricing, Hedging and Optimally Designing Derivatives via Minimization of Risk Measures, *in* R.Carmona, ed., 'Indifference Pricing: Theory and Applications', Princeton University Press, Princeton, USA.

Boonen, T. J. (2015), 'Competitive Equilibria with Distortion Risk Measures', *Astin Bulletin* **45**(3), 703–728.

Borch, K. (1962), 'Equilibrium in a Reinsurance Market', *Econometrica* **30**, 424–444.

Cont, R., R. Deguest & G. Scandolo (2010), 'Robustness and Sensitivity Analysis of Risk Measurement Procedures', *Quantitative Finance* **10**(6), 593–606.

Embrechts, P., H. Liu & R. Wang (2018), 'Quantile-Based Risk Sharing', *Operations Research* **66**(4), 936–949.

European Commission (2009), Directive 2009/138/EC of the European Parliament and of the Council of 25 November 2009 on the taking-up and pursuit of the business of Insurance and Reinsurance (Solvency II), Directives, Official Journal of the European Commission.

Filipovic, D. & G. Svindland (2008), 'Optimal Capital and Risk Allocations for Law- and Cash-Invariant Convex Functions', *Finance and Stochastics* **12**, 423–439.

Filipovic, D. & M. Kupper (2008), 'Equilibrium Prices for Monetary Utility Functions', *International Journal of Theoretical and Applied Finance* **11**, 325–343.

Föllmer, H. & A. Schied (2002), 'Convex Measures of Risk and Trading Constraints', *Finance and Stochastics* **6**(4), 429–447.

Föllmer, H. & A. Schied (2016), *Stochastic Finance - An Introduction in Discrete Time*, 4th rev. edn, de Gruyter, Berlin.

Föllmer, H. & S. Weber (2015), 'The Axiomatic Approach to Risk Measurement for Capital Determination', *Annual Review of Financial Economics* **7**.

Galchion, A. (2010), 'The V@R at Risk', *International Journal of Theoretical and Applied Finance* **13**(4), 503–506.

Jouini, E., W. Schachermayer & N. Touzi (2008), 'Optimal Risk Sharing for Law Invariant Monetary Utility Functions', *Mathematical Finance* **18**(2), 269–292.

Ludkovski, M. & V. Young (2009), 'Optimal Risk Sharing under Distorted Probabilities', *Mathematics and Financial Economics* **2**(2), 87–105.

Raviv, A. (1979), 'The Design of an Optimal Insurance Policy', *American Economic Review* **69**(1), 84–96.

Weber, S. (2018), 'Solvency II, or How to Sweep the Downside Risk Under the Carpet', *Insurance: Mathematics and Economics* **82**, 191–200.

Wilson, R. (1968), 'The Theory of Syndicates', *Econometrica* **36**, 119–132.

A Risk Measures

We denote by $\mathcal{X}$ a vector space of measurable, real-valued functions on a measurable space $(\Omega, \mathcal{F})$ that contains the constants. If $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$, typical examples of $\mathcal{X}$ are L^p-spaces, $p \in [1, \infty]$, where $\mathbb{P}$-almost sure equal functions are identified with each other.

A *monetary risk measure* $\rho : \mathcal{X} \to \mathbb{R}$ is an inverse monotone and cash-invariant function on $\mathcal{X}$:

1. *Inverse Monotonicity:* $X, Y \in \mathcal{X}, X \leq Y \Rightarrow \rho(X) \geq \rho(Y)$
2. *Cash-Invariance:* $X \in \mathcal{X}, m \in \mathbb{R} \Rightarrow \rho(X + m) = \rho(X) - m$

Property 1 states that the risk of a position Y is smaller than the risk of a position X, if the future value of Y is at least X. Property 2 states that risk is measured on a monetary scale: If m Euro are added to X, then the risk of X is exactly reduced by this amount.

In particular, any monetary risk measure corresponds to its *acceptance set*, $\mathcal{A} = \{X \in \mathcal{X} : \rho(X) \leq 0\}$, from which it can be recovered via

$$\rho(X) = \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}\}.$$

Thus, a monetary risk measure can be viewed as a capital requirement: $\rho(X)$ is the minimal capital that has to be added to the position X to make it acceptable.

The choice of a meaningful risk measure for capital regulation is subject to an ongoing discussion between academics and practitioners that began in the mid 1990s. Various desirable properties of monetary risk measures have been proposed, and corresponding classes of risk measures have been identified and characterized. A common requirement in the literature is that diversification should not increase risk. In mathematical terms, diversification corresponds to *quasi-convexity* of ρ, i. e.,

$$\rho(\lambda X + (1 - \lambda)Y) \leq \max\{\rho(X), \rho(Y)\} \tag{11}$$

for $X, Y \in \mathcal{X}$ and $\lambda \in (0, 1)$. In that case, ρ is also a convex functional on $\mathcal{X}$. A monetary risk measure is called a *convex risk measure* if it satisfies condition (11) of quasi-convexity and is hence convex. A convex risk measure is called *coherent* if it is also positively homogeneous, i. e.,

$$\rho(\lambda X) = \lambda \rho(X)$$

for $X \in \mathcal{X}$ and $\lambda \geq 0$. Positive homogeneity is often seen as critical, in particular since the additional concentration risk caused by scaling the financial position is not captured.

A monetary risk measure on a space of random variables $\mathcal{X}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is *distribution-based* (sometimes somewhat misleading also called law-invariant), if $\rho(X) = \rho(Y)$ whenever the distributions of X and Y under $\mathbb{P}$ are equal, i. e., $\mathbb{P}^X = \mathbb{P}^Y$ for $X, Y \in \mathcal{X}$.

Distribution-based risk measures include a wide variety of examples, see, e. g., Föllmer & Schied (2016) and Föllmer & Weber (2015). Throughout this paper, we focus on three prominent examples with different properties:

(a) **Value at risk:** The most commonly used risk measure in practice - and in particular the prescribed risk measure for *Solvency II* purposes - is *value at risk* (V@R). For a given level $\alpha \in (0, 1)$, we denote by $\mathrm{V@R}_\alpha$ the monetary risk measure defined by the acceptance set

$$\mathcal{A}_{\mathrm{V@R}_\alpha} = \{X \in \mathcal{X} | \mathbb{P}[X < 0] \leq \alpha\}. \tag{12}$$

For a financial position X, the value $\mathrm{V@R}_\alpha(X)$ specifies the smallest monetary amount that needs to be added to X such that the probability of a loss becomes smaller than α:

$$\begin{aligned}\mathrm{V@R}_\alpha(X) &= \inf\{m \in \mathbb{R} | \mathbb{P}[X + m < 0] \leq \alpha\} \\ &= -\sup\{c \in \mathbb{R} | \mathbb{P}[X < c] \leq \alpha\} = -q_X^+(\alpha),\end{aligned}$$

where $q^+_X(\alpha)$ is the upper α-quantile of X.

Recall that $\mathrm{V@R}_\alpha$ has two main deficiencies: Firstly, value at risk is not a convex risk measure and may thus penalize diversification beyond the setting of Gaussian or more generally elliptic financial positions. Secondly, $\mathrm{V@R}_\alpha$ neglects extreme losses that occur with small probability. These deficiencies of value at risk were a major reason to develop a systematic theory of coherent and convex risk measures, as initiated by Artzner, Delbaen, Eber & Heath (1999) and Föllmer & Schied (2002).

(b) **Average value at risk:** Another basic example is the *average value at risk* (AV@R), also known as *conditional value at risk*, *tail value at risk*, or *expected shortfall*, which plays a prominent role in the *Swiss Solvency Test*. The average value at risk at level $\beta \in (0,1]$ is defined by

$$\mathrm{AV@R}_\beta(X) := \tfrac{1}{\beta}\int_0^\beta \mathrm{V@R}_\alpha(X)\,d\alpha, \quad X \in \mathcal{X}.$$

In contrast to value at risk, $\mathrm{AV@R}_\beta$ accounts for extreme losses per definition, and it provides incentives for diversification. More precisely, $\mathrm{AV@R}_\beta$ is a coherent measure of risk.

(c) **Range value at risk:** Cont, Deguest & Scandolo (2010) suggest an alternative to V@R and AV@R, called *range value at risk* (RV@R). Letting $\alpha, \beta > 0$ with $\alpha+\beta \leq 1$, they define

$$\mathrm{RV@R}_{\alpha,\beta}(X) = \tfrac{1}{\beta}\int_\alpha^{\alpha+\beta} \mathrm{V@R}_\gamma(X)\,d\gamma, \quad X \in \mathcal{X}.$$

Note that the limiting cases of $\mathrm{RV@R}_{\alpha,\beta}$ correspond to $\mathrm{V@R}_\alpha$ for $\beta \to 0$ and $\mathrm{AV@R}_\beta$ for $\alpha \to 0$. Like V@R, RV@R is a non-convex risk measure, and it may thus penalize diversification.

B Proofs

Proof of Corollary 3.1.

Proof. By Example 2.6, we have $\square_{i=1}^n \mathrm{RV@R}_{\alpha_i,\beta_i}(E_1(\delta)) = \mathrm{RV@R}_{\alpha,\beta}(E_1(\delta))$ for $\alpha = \alpha_1 + \ldots + \alpha_n$, $\beta = \max\{\beta_1, \ldots, \beta_n\}$. It is thus enough to show that

$$\mathrm{RV@R}_{\alpha,\beta}(E_1(\delta)) = -\eta^2(\delta)S_0^2 e^{\mu}\tfrac{1}{\beta}\Big(\Phi(\Phi^{-1}(\alpha+\beta)-\sigma) - \Phi(\Phi^{-1}(\alpha)-\sigma)\Big) - \eta^1(\delta) + \pi. \tag{13}$$

To this end, note first that

$$\begin{aligned}\mathrm{RV@R}_{\alpha,\beta}(E_1(\delta)) &= \mathrm{RV@R}_{\alpha,\beta}(\eta^2(\delta)S_1^2 + \eta^1(\delta) - \pi) \\ &= \eta^2(\delta)\mathrm{RV@R}_{\alpha,\beta}(S_1^2) - \eta^1(\delta) + \pi, \end{aligned}\tag{14}$$

since $\mathrm{RV@R}_{\alpha,\beta}$ is cash-invariant and positively homogeneous. Hence, it remains to compute

$$\mathrm{RV@R}_{\alpha,\beta}(S_1^2) = \tfrac{1}{\beta}\int_\alpha^{\alpha+\beta} \mathrm{V@R}_\gamma(S_1^2)\,d\gamma = \tfrac{1}{\beta}\int_\alpha^{\alpha+\beta} -q_{S_1^2}(\gamma)\,d\gamma.$$

Using the quantile transformation rule for $S_1^2 = f(W_1)$ with the increasing function $f(x) = S_0^2\exp(\mu - \frac{1}{2}\sigma^2 + \sigma x)$ combined with the fact that $q_X(\gamma) = \mathbb{E}[X] + \Phi^{-1}(\gamma)\sigma(X)$ for any normally distributed X, we obtain

$$\begin{aligned}\mathrm{RV@R}_{\alpha,\beta}(S_1^2) &= \tfrac{1}{\beta}\int_\alpha^{\alpha+\beta} -q_{S_1^2}(\gamma)\,d\gamma = \tfrac{1}{\beta}\int_\alpha^{\alpha+\beta} -S_0^2 e^{\mu - \frac{1}{2}\sigma^2 + \sigma q_{W_1}(\gamma)}\,d\gamma \\ &= \tfrac{1}{\beta}\int_\alpha^{\alpha+\beta} -S_0^2 e^{\mu - \frac{1}{2}\sigma^2 + \Phi^{-1}(\gamma)\sigma}\,d\gamma = -S_0^2 e^{\mu - \frac{1}{2}\sigma^2}\tfrac{1}{\beta}\int_\alpha^{\alpha+\beta} e^{\Phi^{-1}(\gamma)\sigma}\,d\gamma.\end{aligned}$$

Substituting $y = \Phi^{-1}(\gamma)$ with $dy = (1/\varphi(\Phi^{-1}(\gamma))d\gamma$ in terms of the density φ of the standard normal distribution leads to

$$\begin{aligned}
\mathrm{RV@R}_{\alpha,\beta}(S_1^2) &= -S_0^2 e^{\mu-\frac{1}{2}\sigma^2}\frac{1}{\beta}\int_{\Phi^{-1}(\alpha)}^{\Phi^{-1}(\alpha+\beta)} e^{\sigma y}\varphi(y)\,dy \\
&= -S_0^2 e^{\mu-\frac{1}{2}\sigma^2}\frac{1}{\beta}\int_{\Phi^{-1}(\alpha)}^{\Phi^{-1}(\alpha+\beta)} e^{\sigma y}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\,dy \\
&= -S_0^2 e^{\mu}\frac{1}{\beta}\int_{\Phi^{-1}(\alpha)}^{\Phi^{-1}(\alpha+\beta)} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y-\sigma)^2}\,dy \\
&= -S_0^2 e^{\mu}\frac{1}{\beta}\int_{\Phi^{-1}(\alpha)-\sigma}^{\Phi^{-1}(\alpha+\beta)-\sigma} \varphi(y)\,dy \\
&= -S_0^2 e^{\mu}\frac{1}{\beta}\Big(\Phi(\Phi^{-1}(\alpha+\beta)-\sigma)-\Phi(\Phi^{-1}(\alpha)-\sigma)\Big).
\end{aligned}$$

Together with (14) this proves (13). Since

$$\mathbb{E}[E_1(\delta)] = \eta^2(\delta)S_0^2 e^{\mu} + \eta^1(\delta) - \pi,$$

the formulae for $\square_{i=1}^n \mathrm{SCR}^i_{\mathcal{A}}(E_1(\delta))$ and $\square_{i=1}^n \mathrm{SCR}^i_{\mathrm{mean}}(E_1(\delta))$, respectively, follow from (2) immediately. □

Proof of Corollary 3.2.

Proof. Recalling that the limiting cases of $\mathrm{RV@R}_{\alpha,\beta}$ correspond to $\mathrm{V@R}_\alpha$ for $\beta \to 0$ and $\mathrm{AV@R}_\beta$ for $\alpha \to 0$, the claim follows from Example 2.6 and Corollary 3.1. More precisely, we have

$$\begin{aligned}
\mathrm{V@R}_\alpha(E_1(\delta)) &= \lim_{\beta\to 0}\mathrm{RV@R}_{\alpha,\beta}(E_1(\delta)) \\
&= \lim_{\beta\to 0}\Big(-\eta^2(\delta)S_0^2 e^{\mu}\frac{1}{\beta}\Big(\Phi(\Phi^{-1}(\alpha+\beta)-\sigma)-\Phi(\Phi^{-1}(\alpha)-\sigma)\Big)-\eta^1(\delta)+\pi\Big) \\
&= -\eta^2(\delta)S_0^2 e^{\mu}\Big(\tfrac{d}{d\beta}\Phi(\Phi^{-1}(\alpha+\beta)-\sigma)\Big)-\eta^1(\delta)+\pi \\
&= -\eta^2(\delta)S_0^2 e^{\mu}\tfrac{\varphi(\Phi^{-1}(\alpha)-\sigma)}{\varphi(\Phi^{-1}(\alpha))}-\eta^1(\delta)+\pi \\
&= -\eta^2(\delta)S_0^2 e^{\mu}\exp(\sigma\Phi^{-1}(\alpha)-\tfrac{\sigma^2}{2})-\eta^1(\delta)+\pi.
\end{aligned}$$

In the same manner, we derive

$$\begin{aligned}
\mathrm{AV@R}_\beta(E_1(\delta)) &= \lim_{\alpha\to 0}\mathrm{RV@R}_{\alpha,\beta}(E_1(\delta)) \\
&= \lim_{\alpha\to 0}\Big(-\eta^2(\delta)S_0^2 e^{\mu}\frac{1}{\beta}\Big(\Phi(\Phi^{-1}(\alpha+\beta)-\sigma)-\Phi(\Phi^{-1}(\alpha)-\sigma)\Big)-\eta^1(\delta)+\pi\Big) \\
&= -\eta^2(\delta)S_0^2 e^{\mu}\tfrac{1}{\beta}\Phi(\Phi^{-1}(\beta)-\sigma)-\eta^1(\delta)+\pi,
\end{aligned}$$

since $\lim_{\alpha\to 0}\Phi^{-1}(\alpha) = -\infty$ and $\lim_{\alpha\to 0}\Phi(\Phi^{-1}(\alpha)-\sigma) = 0$. □