ARE REFERENCE MEASURES OF LAW-INVARIANT FUNCTIONALS UNIQUE?

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Abstract. A functional defined on random variables f is law invariant with respect to a reference probability (or probabilistically sophisticated) if its value only depends on the distribution of its argument f under that measure. In contrast to most of the literature on the topic, we take a concrete functional as given and ask if there can be more than one such reference probability. For wide classes of functionals, we demonstrate that this is not the case *unless* they are (i) constant, or (ii) more generally depend only on the essential supremum and essential infimum of the argument f. Mathematically, the results leverage Lyapunov's Convexity Theorem.

Keywords. Law invariance, probabilistic sophistication, dilatation monotonicity, scenario-based functionals

1. Introduction

Consider a real-valued functional φ defined on a set $\mathcal{D}$ of random variables over a measurable space (Ω, Σ). A probability measure $\mathbb{P}$ on Σ is a *reference measure* for φ if the functional is $\mathbb{P}$-*law invariant*: For all random variables $f, g \in \mathcal{D}$ sharing the same distribution under $\mathbb{P}$, $\varphi(f) = \varphi(g)$. This paper is devoted to the question if the reference measure is unique given φ, and if no, what can be said about the shape of φ.

If $\mathcal{D}$ consists of bounded random variables, the answer to the uniqueness question is seen to be negative. Functions of the essential supremum M and essential infimum m under $\mathbb{P}$ defined by

$$m(f) := \sup\{m \in \mathbb{R} \mid \mathbb{P}(f \leq m) = 0\},$$
$$M(f) := \inf\{m \in \mathbb{R} \mid \mathbb{P}(f \leq m) = 1\},$$

are a counterexample. In fact, whenever $\mathbb{Q}$ is another probability measure equivalent to $\mathbb{P}$ (i.e., an event is $\mathbb{P}$-null if and only if it is $\mathbb{Q}$-null) and two random variables $f, g \in \mathcal{D}$ share the same distribution under $\mathbb{Q}$, then also $M(f) = M(g)$ and $m(f) = m(g)$. In a nutshell, the contribution of this note is to identify large classes of functionals whose reference measures

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are unique *unless* the functional is of this aforementioned shape. More precisely, the following theorem — among others — will be shown.

Theorem. *Suppose φ is a real-valued functional defined on all bounded Σ-measurable random variables. Let $\mathbb{P}$ and $\mathbb{Q}$ be two atomless reference measures of φ.*

(1) *If $\mathbb{P}$ and $\mathbb{Q}$ are not equivalent, then φ is constant.*
(2) *If $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, but not equal, and φ is monotone then $\varphi(f)$ only depends on the quantities $m(f)$ and $M(f)$.*

Related literature. The literature on law-invariant functionals is vast and branches in financial, actuarial, and economic contexts. In the latter, it also trades under the name of "probabilistic sophistication"; cf. Machina & Schmeidler [18], Marinacci [21], Strzalecki [25], and the comments below. We refer to Bellini et al. [2] and Föllmer & Schied [14, Chapter 4] for an overview of studies of law-invariant functionals. The present note follows a more recent strand of literature, particularly on risk measurement and management, which does not take the probability assessment as an underlying primitive, but rather as an *input* that may be varied.

One motivation may be heterogeneous reference beliefs that agents interacting in optimal allocation problems entertain. Given that different financial and insurance companies may use different internal models, heterogeneous probabilistic beliefs enter the picture naturally. To mention a few exemplary publications, each one involving specific classes of functionals to evaluate risk or utility, Boonen [4] studies optimal reinsurance designs under the assumption that insurer and reinsurer use different reference probability models. Boonen & Ghossoub [5] study bilateral risk sharing without aggregate risk and with exposure constraints. Notably though, the two involved reference probability measures may not even display absolute continuity. Arrow's "Theorem of the Deductible" is generalised to situations of heterogeneous beliefs in Ghossoub [15] and later in Chi [8]. Motivated by heterogeneity in internal models, Embrechts et al. [10] give concrete formulae for optimal risk sharing schemes and competitive equilibria involving heterogeneous Value-at-Risk and Expected Shortfall agents. The present manuscript is also not least motivated by Liebrich [16], in particular Theorem 3.3 therein. In view of its goal, the question seems valid if heterogeneity of beliefs "resolves" because reference measures are not unique. For instance, it is conceivable that agents $i = 1, 2$ use reference measures $\mathbb{P}_i$ satisfying $\mathbb{P}_1 \neq \mathbb{P}_2$, but that also $\mathbb{P}_1$ is a reference measure suited to the evaluation functional of 2. Therefore, their interaction could equally be interpreted as a situation of homogeneous beliefs. Our results will fill a gap in the literature by showing that under mild conditions typically fulfilled in the aforementioned problem setting, this is not the case. Belief heterogeneity is therefore *bona fide* heterogeneity.

In another related set of contributions, the fixed underlying reference measure is replaced by a whole set $\mathcal{Q}$ thereof. This is motivated by situations in which relevant distributional properties of financial payoffs depend on the occurrence (or absence) of economic shock scenarios. Exemplary is Wang & Ziegel [27], where law invariance of risk measures is generalised to

scenario-basedness: Whenever two arguments f and g agree in distribution under each measure $\mathbb{Q}$ in a finite set $\mathcal{Q}$ of probabilistic economic scenarios, then the risk of f equals the risk of g. Theoretically related, Shen et al. [24] studies the richness of the *range of distributions* that can be obtained by fixing a random variable f, but varying within a finite set $\mathcal{Q}$ the reference measure $\mathbb{Q}$ under which the distribution of f is evaluated.
Third, this note is motivated by the phenomenon of *uncertainty*, in which the (probabilistic) parameters governing an economic outcome cannot be determined with full precision. There is endless literature on this topic. One work that we would like to draw attention to is Fadina et al. [12], axiomatically studying generalised risk measures whose input is not only a random variable modelling a payoff profile, but a pair also comprising a set of scenarios — atomless probability measures $\mathbb{Q}$ on the underlying measurable space. These offer varying probabilistic mechanisms of scenario realisation whose relevance may not be discounted because of uncertainty. Such a point of view requires a more detailed and diversified understanding of law invariance. The notion is therefore fanned out threefold in [12, Section 4]. Against this backdrop, the uniqueness results in the present note could be read as singling out situations in which law invariance and uncertainty are irreconcilable: the reference measure cannot be uncertain.[1]

This leads to another important aspect: the overlooked differences between law invariance and probabilistic sophistication. In their introduction of probablistic sophistication, Machina & Schmeidler [18] depart from Savage's foundation to Bayesian statistics [23]. Therein, subjective probabilities are represented by *finitely additive probability charges* that are shown to be convex-ranged; cf. [18, Theorems 1–3]. The latter is the proper finitely additive analogue of an atomless probability measure and means that the probability assessment is maximally informative and displays a "fractal-like" structure: conditional on any event with positive probability, binary bets with any conceivable odds are available. In contrast, the reference measure under which law invariance is observed is *countably additive*. Finite additivity is the default in many applications because behavioural axioms guaranteeing countable additivity tend to be very strong; see, e.g., the discussions in Fine [13, Section 3] and [18, Section 6.2]. A second and more important distinction is that probabilistic sophistication requires the reference measure to be *unique*. No such statement is made when defining law invariance, where the reference measure is typically externally given as a primitive of the problem in question or a feature of the underlying state space. An important point of comparison in the literature on probabilistic sophistication is Chew & Sagi [7], whose very general approach and uniqueness result share the spirit of the present manuscript. A detailed discussion is provided in Remark 5.10.

We shall bridge the gap between law invariance and probabilistic sophistication by introducing $\mathfrak{p}$-*invariance* of a functional φ, where $\mathfrak{p}$ is a (typically convex-ranged) reference probability. It is

[1] This point should not be mistaken for the argument of Epstein [11] who identifies probabilistic sophistication — the existence of a unique reference measures — with uncertainty neutrality: "Such exclusive reliance on probabilities is, in particular, inconsistent with the typical 'uncertainty averse' behaviour exhibited in Ellsberg-type experiments. Thus it is both intuitive and consistent with common practice to identify probabilistic sophistication with uncertainty neutrality." ([11, p. 585])

crucial to appreciate the importance of a convex range, allowing to seamlessly identify a setting of Savage acts (random variables) with lotteries (probability distributions over real outcomes). Without, nearly all interesting implications of law invariance and probabilistic sophistication tend to fail. This is underscored for our purposes by Examples 3.4 and 4.7 below. Moreover, dealing with the lack of countable additivity of probability assessments tends to be tedious, but is facilitated by convex-rangedness. In total, putting a particular focus on the existence of convex-ranged reference probabilities as done in this paper seems to be fully warranted.

Outline and contributions of the paper. In the results outside of Section 6, the domain of definition of the functionals under consideration will be the Banach $\mathcal{X}$ space of all bounded random variables or the subset $\mathcal{X}_0$ of all simple random variables. We first prove in Section 3 — imposing very mild conditions on the functional φ where necessary — that convex-ranged reference probabilities have to be equivalent unless φ is constant. This admits to focus the attention in the sequel on the case of equivalent reference measures. In Section 4, we prove what is perhaps the main result of the paper, Theorem 4.1. It formulates, as already advertised above, that monotone functionals $\varphi\colon \mathcal{X}_0 \longrightarrow \mathbb{R}$ *either* have at most one convex-ranged (i.e., fully informative) reference probability *or* display the special shape $\varphi = T \circ (m, M)$ for a function $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ and the essential infimum m and supremum M. This general dichotomy is then extended and varied in multiple situations in Section 5, namely to:

- monotone lower semicontinuous (l.s.c.) functionals on the larger space $\mathcal{X}$ (Corollary 5.1).
- monotone functionals with the Fatou property in Corollary 5.3, in which case one even observes no dependence of φ on m: $\varphi = \varphi \circ M$.
- Dropping monotonicity of φ, for dilatation monotone and l.s.c. φ in Theorem 5.8.

Notably, we keep the structural properties imposed on φ as minimal as possible and make no use of convexity.

The aforementioned dichotomy becomes sharper if the functional φ is more regular with respect to the pointwise order. This allows us to also consider spaces of unbounded random variables in Section 6. In these cases, we verify like in the nonequivalent case that either the reference probability is unique, or the functional φ must be constant; cf. Proposition 5.4 and Corollary 6.2.

Mathematically, the results mostly leverage Lyapunov's Convexity Theorem and rely on a refined, but more technical version of Theorem 4.1, Proposition 4.6. In Section 7 we generalise this mathematical strategy to obtain a parallel result comparing $\mathfrak{p}$-invariance to the much broader property of scenario-basedness introduced by [27]. This is facilitated by paying the cost of imposing more structure on φ; we shall consider the example of Choquet integrals.

We illustrate sharpness of almost all of our results with counterexamples.

2. Preliminaries

Throughout this paper, (Ω, Σ) denotes a measurable space, $\mathcal{X}$ the (real) vector space of all bounded Σ-measurable random variables with real values, and $\mathcal{X}_0 \subset \mathcal{X}$ the subspace of all simple random variables, i.e., each $f \in \mathcal{X}_0$ attains only finitely many different values. $\mathcal{X}$ is a Banach space when equipped with the supremum norm defined by $\|f\|_\infty := \sup_{\omega\in\Omega} |f(\omega)|$. This setting is "model free"; no probability measure is present.

Throughout, probabilities are either measured by finitely additive probability charges on (Ω, Σ) (denoted by $\mathfrak{p}$ and $\mathfrak{q}$) or countably additive probability measures denoted by $\mathbb{P}$ and $\mathbb{Q}$, respectively. We say that:

(a) $\mathfrak{q}$ is *absolutely continuous* with respect to $\mathfrak{p}$ ($\mathfrak{q} \ll \mathfrak{p}$) if each $\mathfrak{p}$-null set is $\mathfrak{q}$-null.[2]
(b) $\mathfrak{p}$ and $\mathfrak{q}$ are *equivalent* ($\mathfrak{p} \approx \mathfrak{q}$) if $\mathfrak{q} \ll \mathfrak{p}$ and $\mathfrak{p} \ll \mathfrak{q}$.
(c) $\mathfrak{p}$ is *convex-ranged* if, for every $A \in \Sigma$, the set $\{\mathfrak{p}(B) \mid B \in \Sigma,\ B \subset A\}$ is a convex subset of $[0, 1]$.

Combining [3, Corollary 5.3.3 & Theorem 11.4.5], convex-ranged probabilities exist on every infinite σ-algebra. Recall from [3, Theorem 5.1.6] that a probability measure $\mathbb{P}$ is convex-ranged if and only if it is *atomless*, i.e., for all $A \in \Sigma$ with $\mathbb{P}(A) > 0$ one finds a subset $B \in \Sigma$ such that $0 < \mathbb{P}(B) < \mathbb{P}(A)$.

Given $f, g \in \mathcal{X}$ and a probability charge $\mathfrak{p}$, $f \sim_{\mathfrak{p}} g$ denotes "equality in distribution" under $\mathfrak{p}$, i.e., for all intervals $I \subset \mathbb{R}$, $\mathfrak{p}(f \in I) = \mathfrak{p}(g \in I)$. Given that countably additive measures are fully determined by their values on generating π-systems, $f \sim_{\mathbb{P}} g$ holds for a probability measure $\mathbb{P}$ if and only if the induced Borel probability measures $\mathbb{P} \circ f^{-1}$ and $\mathbb{P} \circ g^{-1}$ on the real line agree. The following result is well known (see, e.g., [14, Lemma A.23]).

Lemma 2.1. *If $\mathbb{P}$ is an atomless probability measure, the set $\{\mathbb{P} \circ f^{-1} \mid f \in \mathcal{X}\}$ of potential distributions realised under $\mathbb{P}$ by elements of $\mathcal{X}$ contains all Borel probability measures μ on $\mathbb{R}$ with compact support.*

Given a probability charge $\mathfrak{p}$ on Σ, the $\mathfrak{p}$-*essential range* correspondence is the map $\mathcal{W}_{\mathfrak{p}} \colon \mathcal{X}_0 \longrightarrow 2^{\mathbb{R}}$ defined by

$$\mathcal{W}_{\mathfrak{p}}(f) := \{x \in \mathbb{R} \mid \mathfrak{p}(f = x) > 0\}.$$

Moreover, for a conditioning event $C \in \Sigma$ with $\mathfrak{p}(C) > 0$, the conditional probability charge $\mathfrak{p}^C \colon \Sigma \longrightarrow [0, 1]$ is defined by $\mathfrak{p}^C(A) := \frac{\mathfrak{p}(A\cap C)}{\mathfrak{p}(C)}$.

Let $\mathcal{D} \subset \mathcal{X}$ be the domain of definition of a functional $\varphi \colon \mathcal{D} \longrightarrow \mathbb{R}$. φ is called:

(a) *monotone* if, whenever $f, g \in \mathcal{D}$ satisfy $f(\omega) \le g(\omega)$ for all $\omega \in \Omega$, then $\varphi(f) \le \varphi(g)$.
(b) *lower semicontinuous* (l.s.c.) if, for each $c \in \mathbb{R}$, the lower level set $\{f \in \mathcal{X} \mid \varphi(f) \le c\}$ is closed with respect to $\|\cdot\|_\infty$.

Given a probability charge $\mathfrak{p}$, we say that:

(c) f is *invariant under* $\mathfrak{p}$ or $\mathfrak{p}$-*invariant* if, for all $f, g \in \mathcal{D}$,

$$f \sim_{\mathfrak{p}} g \quad \Longrightarrow \quad \varphi(g) = \varphi(g). \tag{2.1}$$

The set of all probability charges $\mathfrak{p}$ satisfying (2.1) for φ is called the set of *reference probabilities* for φ and denoted by $\mathfrak{Ref}(\varphi)$.

[2] In the terminology of [3], this is only "weak absolute continuity"; cf. [3, p. 159].

3. Reference measures have to be equivalent

Our first result asserts that if a nontrivial functional φ on $\mathcal{X}_0$ has two convex-ranged reference probabilities $\mathfrak{p}, \mathfrak{q}$, then they necessarily share the same null events.

Proposition 3.1. *Suppose $\mathfrak{p} \not\approx \mathfrak{q}$ are two nonequivalent convex-ranged probability charges on (Ω, Σ). Then the following are equivalent for a functional $\varphi\colon \mathcal{X}_0 \longrightarrow \mathbb{R}$.*

1. *$\{\mathfrak{p}, \mathfrak{q}\} \subset \mathfrak{Ref}(\varphi)$*
2. *φ is constant.*

Proof. It is trivial to see that (2) implies (1), and we can focus on the converse implication. Let $\mathfrak{p}, \mathfrak{q} \in \mathfrak{Ref}(\varphi)$ be convex-ranged and assume that $\mathfrak{p} \not\approx \mathfrak{q}$. Without loss, we may assume that there is an event $N \in \Sigma$ with $\mathfrak{p}(N) > 0$ and $\mathfrak{q}(N) = 0$. Using that $\mathfrak{p}$ is convex-ranged, we can even assume $\mathfrak{p}(N) = \frac{1}{r}$ for some $r \in \mathbb{N}$ and complete N to a measurable partition of Ω into events $A_1, \dots, A_r \in \Sigma$ such that $A_1 = N$ and $\mathfrak{p}(A_i) = \frac{1}{r}$, $1 \le i \le r$. Moreover, observe that for all $h \in \mathcal{X}_0$, $h \sim_{\mathfrak{q}} h\mathbf{1}_{N^c}$, whence

$$\varphi(h) = \varphi(h\mathbf{1}_{N^c}). \tag{3.1}$$

Fix $f \in \mathcal{X}_0$. For an arbitrary $2 \le i \le r$ set $\mathcal{W}_i := \{x \in \mathbb{R} \mid \mathfrak{p}(\{f = x\} \cap A_i) > 0\}$. Use convex-rangedness of $\mathfrak{p}$ once more and partition $A_1 = N$ into events $(B_x)_{x \in \mathcal{W}_i}$ satisfying $\mathfrak{p}(B_x) = \mathfrak{p}(\{f = x\} \cap A_i)$ for all $x \in \mathcal{W}_i$. By construction, the random variable

$$g := \sum_{x \in \mathcal{W}_i} x\mathbf{1}_{B_x} + f\mathbf{1}_{(A_1 \cup A_i)^c}$$

satisfies $g \sim_{\mathfrak{p}} f\mathbf{1}_{N^c}$ and $g\mathbf{1}_{N^c} = f\mathbf{1}_{(A_1 \cup A_i)^c}$. Together with $\mathfrak{p}$-invariance of φ and (3.1), we conclude that

$$\varphi(f) = \varphi(f\mathbf{1}_{N^c}) = \varphi(g) = \varphi(g\mathbf{1}_{N^c}) = \varphi(f\mathbf{1}_{(A_1 \cup A_i)^c}). \tag{3.2}$$

Using identity (3.2) iteratively, $\varphi(f) = \varphi(0)$. ∎

Remark 3.2.

1. The proof of Proposition 3.1 shows a finer observation. If a functional $\varphi\colon \mathcal{X}_0 \longrightarrow \mathbb{R}$ has a convex-ranged reference probability $\mathfrak{p}$, then either $\mathfrak{p}$ is minimal in that $\mathfrak{p} \ll \mathfrak{q}$ for all $\mathfrak{q} \in \mathfrak{Ref}(\varphi)$, or φ is constant.
2. On a more technical note, Proposition 3.1 only requires the underlying structure of an algebra of events as is, for instance, assumed in [7].

An unfortunate aspect of Proposition 3.1 is that the functional φ is defined only on $\mathcal{X}_0$. We can say more about the larger domain of definition $\mathcal{X}$ if we additionally impose mild restrictions.

Corollary 3.3. *Let $\varphi\colon \mathcal{X} \longrightarrow \mathbb{R}$ be a functional.*

1. *If φ is monotone or continuous, then Proposition 3.1 holds* verbatim.
2. *If $\mathfrak{Ref}(\varphi)$ contains two atomless reference measures $\mathbb{P} \not\approx \mathbb{Q}$, then φ is constant.*

Proof. To prove assertion (1), suppose that we find two nonequivalent convex-ranged reference probabilities $\mathfrak{p}, \mathfrak{q} \in \mathfrak{Ref}(\varphi)$. Then Proposition 3.1 reveals $\varphi|_{\mathcal{X}_0} \equiv \varphi(0)$. This clearly implies constancy of φ under the assumption of continuity. Under monotonicity, it suffices to observe for every $f \in \mathcal{X}$ that $\varphi(0) = \varphi(-\|f\|_\infty) \leq \varphi(f) \leq \varphi(\|f\|_\infty) = \varphi(0)$.

The verification of assertion (2) is analogous to the proof of Proposition 3.1, replacing $\mathfrak{p}$ by $\mathbb{P}$ and $\mathfrak{q}$ by $\mathbb{Q}$. The only difference is the following argument. Let $f \in \mathcal{X}$ be arbitrary. Lemma 2.1 admits to select a random variable $h_i\colon A_1 \longrightarrow \mathbb{R}$ such that $\mathbb{P}^{A_1} \circ h_i^{-1} = \mathbb{P}^{A_i} \circ (f|_{A_i})^{-1}$. Consequently, $f\mathbf{1}_{N^c} \sim_{\mathbb{P}} h_i\mathbf{1}_{A_1} + f\mathbf{1}_{(A_1\cup A_i)^c}$, and one concludes

$$\varphi(f) = \varphi\big(h_i\mathbf{1}_{A_1} + f\mathbf{1}_{(A_1\cup A_i)^c}\big) = \varphi(f\mathbf{1}_{(A_1\cup A_i)^c}).$$

□

In parallel with the above remark, the proof of Corollary 3.3(2) shows a finer property: Every atomless reference measure $\mathbb{P}$ of a nonconstant functional $\varphi\colon \mathcal{X} \longrightarrow \mathbb{R}$ must satisfy $\mathbb{P} \ll \mathfrak{q}$ for all reference probabilities $\mathfrak{q} \in \mathfrak{Ref}(\varphi)$.

Proposition 3.1 should not be misunderstood; a nonconstant functional $\varphi\colon \mathcal{X}_0 \longrightarrow \mathbb{R}$ can have nonequivalent reference probabilities. More precisely, convex-rangedness is necessary for the assertion to hold.

Example 3.4. Let $\mathfrak{p}$ be any convex-ranged Borel probability charge on $\mathbb{R}$ satifying $\mathfrak{p}(\mathbb{R}_+) = 1$ set $\mathfrak{q}_x := \frac{1}{3}\mathfrak{p} + \frac{2}{3}\delta_x$, where δ_x denotes the Dirac delta concentrated at $x < 0$. Consider the $\mathfrak{p}$-invariant functional $\varphi\colon \mathcal{X}_0 \longrightarrow \mathbb{R}$ defined by

$$\varphi(f) = \int_{\mathbb{R}} f \, \mathrm{d}\mathfrak{p}.$$

One observes that $f, g \in \mathcal{X}_0$ satisfy $f \sim_{\mathfrak{q}_x} g$ if and only if $f \sim_{\mathfrak{p}} g$ and $f(x) = g(x)$. In particular, $\{\mathfrak{q}_x \mid x < 0\} \subset \mathfrak{Ref}(\varphi)$.

4. Uniqueness for functionals on simple functions

The previous section uncovered that convex-ranged reference probabilities of nonconstant functionals φ have to be equivalent. Throughout the remainder of the paper, we shall therefore focus on *equivalent* reference probabilities. One should keep in mind that if $\mathbb{P}$ is a convex-ranged (equivalently, atomless) probability measure and $\mathbb{Q} \approx \mathbb{P}$, then $\mathbb{Q}$ is also convex-ranged. This implication does not necessarily hold if $\mathbb{P}$ and $\mathbb{Q}$ are replaced by finitely additive probability charges $\mathfrak{p}$ and $\mathfrak{q}$. Whenever we speak of equivalent and convex-ranged reference probabilities in the following, this may therefore pose some restriction.

In order to formulate the versatile result at the core of all further discussions, we also need to introduce two functionals m and M on $\mathcal{X}$ that could be called the "worst-case" and "best-case risk measure". Given a probability charge $\mathfrak{p}$ on (Ω, Σ) and any random variable $f\colon \Omega \longrightarrow \mathbb{R}$, we set

$$m(f) := \sup\{m \in \mathbb{R} \mid \mathfrak{p}(f \leq m) = 0\},$$
$$M(f) := \inf\{m \in \mathbb{R} \mid \mathfrak{p}(f \leq m) = 1\}.$$

As usual, $\sup \varnothing = -\inf \varnothing = -\infty$. m and M are $\mathfrak{p}$-invariant and their definition only depends on the null sets of said probability. Departing from an equivalent probability charge $\mathfrak{q} \approx \mathfrak{p}$

therefore leads to the same values $m(f)$ and $M(f)$, and both functionals are also $\mathfrak{q}$-invariant. In short,

$$\mathfrak{Ref}(m) = \mathfrak{Ref}(M) = \{\mathfrak{q} \mid \mathfrak{q} \approx \mathfrak{p}\}.$$

As all probability measures appearing in the sequel will be equivalent, our notation suppresses dependence on the measure.

We are now ready to state the main result of this section: A *monotone* functional φ on $\mathcal{X}_0$ has at most one convex-ranged reference probability unless $\varphi(f)$ is fully determined by the quantities $m(f)$ and $M(f)$.

Theorem 4.1. *Let $\mathfrak{p} \approx \mathfrak{q}$ be two convex-ranged probability charges on Σ. Suppose a monotone functional $\varphi\colon \mathcal{X}_0 \longrightarrow \mathbb{R}$ satisfies $\{\mathfrak{p}, \mathfrak{q}\} \subset \mathfrak{Ref}(\varphi)$. Then one of the following alternatives holds:*

(1) $\mathfrak{p} = \mathfrak{q}$.
(2) $\varphi = T \circ (m, M)$ *for a function $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$.*

Before we proceed with the proof of Theorem 4.1, we observe that the assumption of monotonicity is not too restrictive if the goal is its application in areas like utility evaluation or risk measurement, but a necessary assumption nevertheless. More dramatically, the result fails even if $\varphi\colon \mathcal{X}_0 \longrightarrow \mathbb{R}$ is continuous. The finer Proposition 4.6 below will demonstrate in which sense Theorem 4.1 generalises to nonmonotone functionals.

Example 4.2. Construct a functional $\varphi\colon \mathcal{X}_0 \longrightarrow \mathbb{R}$ in the following manner. For $f \in \mathcal{X}_0$ let $x_1 < \ldots < x_n$ be the different values in $\mathcal{W}_{\mathfrak{p}}(f)$. We set $\varphi(f) = 0$ if $\mathcal{W}_{\mathfrak{p}}(f)$ is a singleton and, else,

$$\varphi(f) = \max_{1 \le i \le n-1} (x_{i+1} - x_i).$$

We have $\mathfrak{q} \in \mathfrak{Ref}(\varphi)$ whenever $\mathfrak{q} \approx \mathfrak{p}$. Moreover, φ is continuous. To see the latter, note that for every $f \in \mathcal{X}_0$ and $x_1 < \ldots < x_n$ as above, the "cumulative distribution function" of f under $\mathfrak{p}$ can be defined by

$$F_f(x) := \mathfrak{p}(f \le x) = \sum_{i=1}^{n} \mathfrak{p}(f = x_i) \mathbf{1}_{[x_i, \infty[}(x), \quad x \in \mathbb{R}.$$

F_f is nondecreasing and right-continuous. Its left-continuous generalised inverse function is given by

$$F_f^{-1}(s) := \inf\{x \in \mathbb{R} \mid F_f(x) \ge s\}, \quad s \in]0, 1[,$$

and the associated right-hand limit by $F_f^{-1}(s+) := \inf_{t>s} q_f(t)$, $s \in]0, 1[$. The finitely many discontinuities s of F_f^{-1} at which $F_f^{-1}(s) < F_f^{-1}(s+)$ holds are given precisely by the x_i's. Thus, for all $f \in \mathcal{X}_0$,

$$\varphi(f) = \max_{s \in]0,1[} \left(F_f^{-1}(s+) - F_f^{-1}(s)\right).$$

One verifies that, for $f, g \in \mathcal{X}$ and $s \in]0, 1[$, we have

$$F_f^{-1}(s) - \|f - g\|_\infty \le F_g^{-1}(s) \le F_f^{-1}(s) + \|f - g\|_\infty,$$

which means that, for all $f, g \in \mathcal{X}_0$,

$$\begin{aligned}\varphi(f) - 2\|f-g\|_\infty &= \sup_{s\in]0,1[} \big(F_f^{-1}(s+) - F_f^{-1}(s)\big) - 2\|f-g\|_\infty \\ &\le \sup_{s\in]0,1[} \big(F_g^{-1}(s+) - F_g^{-1}(s)\big) = \varphi(g) \\ &\le \sup_{s\in]0,1[} \big(F_f^{-1}(s+) - F_f^{-1}(s)\big) + 2\|f-g\|_\infty = \varphi(f) + 2\|f-g\|_\infty.\end{aligned}$$

The estimate entails that, for all $f, g \in \mathcal{X}_0$, $|\varphi(f) - \varphi(g)| \le 2\|f-g\|_\infty$, that is, φ is 2-Lipschitz continuous.

However, φ fails alternative (2) in Theorem 4.1. Indeed, let A, B, C be three pairwise disjoint events, each with positive $\mathfrak{p}$-probability. Then $f = \mathbf{1}_A - \mathbf{1}_B$ and $g := \mathbf{1}_A - \mathbf{1}_{B\cup C}$ satisfy $m(f) = m(g) = -1$, $M(f) = M(g) = 1$, but

$$\varphi(f) = 1 \ne 2 = \varphi(g).$$

4.1. **Proof of Theorem 4.1.** The proof of Theorem 4.1 rests on a number of auxiliary results. The first one is the version of Lyapunov's Convexity Theorem for finitely additive probability charges from [3].

Lemma 4.3. *Let $\mathfrak{p}, \mathfrak{q}$ be two convex-ranged probability charges on (Ω, Σ). Then the set*

$$\mathcal{R} := \big\{\big(\mathfrak{p}(A), \mathfrak{q}(A)\big) \mid A \in \Sigma\big\} \tag{4.1}$$

is convex.

Proof. By [3, Remark 5.1.7 & Theorem 11.4.5], both $\mathfrak{p}$ and $\mathfrak{q}$ are *strongly continuous* in the sense of [3, Definition 5.1.4]. By [3, Theorem 11.4.9], $\mathcal{R}$ is convex. ☐

Lemma 4.4. *Suppose $\mathfrak{p} \approx \mathfrak{q}$ are convex-ranged probability charges on Σ such that $\mathfrak{p} \ne \mathfrak{q}$ and define $\mathcal{R}$ by (4.1). Let $p_1 \in]0,1[$ be arbitrary. Then there exists a nondecreasing sequence $(p_n)_{n\in\mathbb{N}} \subset]0,1]$ such that $\{(p_n, p_{n+1}) \mid n \in \mathbb{N}\} \subset \mathcal{R}$ and $\sup_{n\in\mathbb{N}} p_n = 1$.*

Proof. The set $\mathcal{R}$ is convex by Lemma 4.3 and therefore contains the diagonal in $[0,1]^2$. $\mathfrak{p} \ne \mathfrak{q}$ implies that we can find an event $A_\star \in \Sigma$ such that the set $\{(\mathfrak{p}(A_\star), \mathfrak{q}(A_\star)), (\mathfrak{p}(A_\star^c), \mathfrak{q}(A_\star^c))\}$ consists of one point strictly above and one point strictly below the diagonal. Thus, again by convexity of $\mathcal{R}$, for all $p \in]0,1[$ there is $s > 0$ such that $(p, p \pm s) \in \mathcal{R}$. Consequently, the sets

$$\mathcal{I}_p := \{q \in [0,1] \mid (p,q) \in \mathcal{R}\}$$

are intervals with interior point p.

Having constructed, $p_1, \dots, p_n$ with the desired properties, let $\widehat{p} := \sup \mathcal{I}_{p_n}$. If $\widehat{p} = p_n$, then $p_n = 1$ holds and we can set $p_k = 1$ for all $k \ge n$. Else, we may select p_{n+1} satisfying $\frac{1}{2}(p_n + \widehat{p}) < p_{n+1} < \widehat{p}$, which must therefore satisfy $(p_n, p_{n+1}) \in \mathcal{R}$.

Let $p := \lim_{n\to\infty} p_n$ and assume for contradiction that $p < 1$. Then we can find $q > p$ such that $(p, q) \in \mathcal{R}$. Select $B \in \Sigma$ such that $\mathfrak{p}(B) = p$ and $\mathfrak{q}(B) = q$. Using that $\mathfrak{q}$ has convex range, we can find an increasing sequence $(B_k)_{k\in\mathbb{N}}$ of events $B_k \subset B$ such that $\mathfrak{q}(B_k) = (1 - 2^{-k})q$, $k \in \mathbb{N}$. In particular, $\mathfrak{q}(B \setminus B_k) > 0$ implies $\mathfrak{p}(B \setminus B_k) > 0$. Therefore, for n large enough,

$p_n \geq \mathfrak{p}(B_k)$ has to hold. In turn, using convex-rangedness of $\mathfrak{q}$ once more,

$$\begin{aligned} p_{n+1} &> \tfrac{1}{2}\big(p_n + \sup \mathcal{I}_{p_n}\big) \\ &\geq \tfrac{1}{2}\big(p_n + \sup\{\mathfrak{q}(C) \mid C \in \Sigma,\, \mathfrak{p}(C) = p_n,\, C \supset B_k\}\big) \\ &\geq \tfrac{1}{2}\big(p_n + (1-2^{-k})q\big). \end{aligned}$$

Taking the limit as $n \to \infty$ and rearranging the inequality, $p \geq (1-2^{-k})q$. Taking the limit $k \to \infty$, $p \geq q$, a contradiction. Thus, $p = 1$ has to hold. $\square$

Lemma 4.5. *Suppose $\mathfrak{p} \approx \mathfrak{q}$ are convex-ranged probability charges on Σ such that $\mathfrak{p} \neq \mathfrak{q}$. Then for all $\pi \in]0,1[$ there is $C \in \Sigma$ with $\mathfrak{p}(C) = \pi$ and $\mathfrak{p}^C \neq \mathfrak{q}^C$.*

Proof. Select $0 < r < \pi$ and $A \in \Sigma$ such that $r = \mathfrak{p}(A) < \mathfrak{q}(A)$. Consider the measurable space $(A^c, \Sigma \cap A^c)$ as well as the convex-ranged conditional probability charges $\mathfrak{p}^{A^c}$ and $\mathfrak{q}^{A^c}$. Invoking Lemma 4.3, we find $B \in \Sigma$, $B \subset A^c$, such that $\mathfrak{p}^{A^c}(B) = \mathfrak{q}^{A^c}(B) = \frac{\pi - r}{1-r}$. Set $C := A \cup B$ and note that $\mathfrak{p}(C) = \pi$. Moreover, we compute

$$\mathfrak{p}^C(A) = \tfrac{r}{\pi} \quad \text{and} \quad \mathfrak{q}^C(A) = \frac{\mathfrak{q}(A)(1-r)}{\pi - r + \mathfrak{q}(A)(1-\pi)}.$$

Interpreting $\mathfrak{q}^C(A)$ as function in the variable $\mathfrak{q}(A)$, one has at most one solution for $\mathfrak{p}^C(A) = \mathfrak{q}^C(A)$. Thus, choosing $\mathfrak{q}(A)$ suitably, one can guarantee that $\mathfrak{p}^C(A) \neq \mathfrak{q}^C(A)$ and that therefore $\mathfrak{p}^C \neq \mathfrak{q}^C$. $\square$

The following Proposition 4.6 can be interpreted as a finer version of Theorem 4.1 which does not depend on monotonicity. It shows for a nonconstant functional φ on $\mathcal{X}_0$ whose reference probabilities $\mathfrak{Ref}(\varphi)$ contain more than one convex-ranged charge that every pair of functions $\mathcal{X}_0$ with the same essential ranges are mapped to the same number under φ.

Proposition 4.6. *Suppose $\mathfrak{p} \approx \mathfrak{q}$ are convex-ranged probability charges on Σ such that $\mathfrak{p} \neq \mathfrak{q}$. Assume furthermore that $\varphi\colon \mathcal{X}_0 \longrightarrow \mathbb{R}$ satisfies $\{\mathfrak{p}, \mathfrak{q}\} \subset \mathfrak{Ref}(\varphi)$. Then, for all $f, g \in \mathcal{X}_0$,*

$$\mathcal{W}_{\mathfrak{p}}(f) = \mathcal{W}_{\mathfrak{p}}(g) \quad \Longrightarrow \quad \varphi(f) = \varphi(g).$$

In particular, a nonconstant functional φ on $\mathcal{X}_0$ has at most one convex-ranged reference probability $\mathfrak{p}$ or $\mathfrak{Ref}(\varphi)$ contains all $\mathfrak{q} \approx \mathfrak{p}$.

Proof. Recall the definition of the set $\mathcal{R}$ in (4.1). We prove the assertion by induction over $n := |\mathcal{W}_{\mathfrak{p}}(f)|$.
$\underline{n = 1}$: This is clear.
$\underline{n \to n+1}$: Let $x_1, \dots, x_{n+1}$ be an enumeration of $\mathcal{W}_{\mathfrak{p}}(f) = \mathcal{W}_{\mathfrak{p}}(g)$. In a first step, we shall assume that

$$\mathfrak{p}(f = x_1) = \mathfrak{p}(g = x_1)$$

fails and cannot even be achieved by permuting the enumeration. Without loss, assume that $p_1 := \mathfrak{p}(f = x_1) < \mathfrak{p}(g = x_1)$ and set $f_1 := f$. Let $(p_k)_{k \in \mathbb{N}}$ be the sequence of probabilities

constructed in Lemma 4.4. An inspection of the proof allows us to select the sequence such that
$$N := \min\{k \in \mathbb{N} \mid p_k = \mathfrak{p}(g = x_1)\} < \infty.$$
Iteratively, for all $1 \leq i \leq N-1$ we define random variables f_{i+1} satisfying $\mathcal{W}_\mathfrak{p}(f_{i+1}) = \mathcal{W}_\mathfrak{p}(f)$, $\varphi(f_{i+1}) = \varphi(f)$, and $\mathfrak{p}(f_i = x_1) = p_{i+1}$ in the following manner.
As $(p_i, p_{i+1}) \in \mathcal{R}$, we may select $A \in \Sigma$ such that $\mathfrak{p}(A) = p_i$ and $\mathfrak{q}(A) = p_{i+1}$. Using that $\mathfrak{p}$ has convex range, we can select $v \sim_\mathfrak{p} f_i$ such that $\{v = x_1\} = A$. Next, select $B \in \Sigma$ such that $\mathfrak{p}(B) = \mathfrak{q}(B) = p_{i+1}$. Using convex-rangedness of $\mathfrak{q}$ this time, we find $f_{i+1} \in \mathcal{X}_0$ such that $f_{i+1} \sim_\mathfrak{q} v$ and $\{f_{i+1} = x_1\} = B$. Note that $\varphi(f_{i+1}) = \varphi(v) = \varphi(f_i) = \varphi(f)$ and continue.
The preceding construction yields $f_N \in \mathcal{X}_0$ with the property $\varphi(f_N) = \varphi(f)$, $\mathcal{W}_\mathfrak{p}(f_N) = \mathcal{W}_\mathfrak{p}(f)$, and $\mathfrak{p}(f_N = x_1) = \mathfrak{p}(g = x_1)$. Replace f above by f_N and define $\pi := \mathfrak{p}(f_N \neq x_1) = \mathfrak{p}(g \neq x_1)$. By Lemma 4.5 we find $C \in \Sigma$ such that $\mathfrak{p}(C) = \pi$ and $\mathfrak{p}^C \neq \mathfrak{q}^C$ on the trace σ-algebra $\mathcal{G} := \Sigma \cap C$.
Let $\mathcal{Y}$ be the space of simple $\mathcal{G}$-measurable random variables defined on C. Using convex-rangedness of $\mathfrak{p}$, we can find $u \sim_\mathfrak{p} f_N$ and $v \sim_\mathfrak{p} g$ such that $\{u = x_1\} = \{v = x_1\} = C^c$. The functional $\psi\colon \mathcal{Y} \longrightarrow \mathbb{R}$ defined by
$$\psi(h) := \varphi(h\mathbf{1}_C + x_1\mathbf{1}_{C^c})$$
is verified to be both $\mathfrak{p}^C$- and $\mathfrak{q}^C$-invariant. By induction hypothesis and $\mathfrak{p}$-probabilistic sophistication,
$$\varphi(f) = \varphi(f_N) = \varphi(u) = \psi(u\mathbf{1}_C) = \psi(v\mathbf{1}_C) = \varphi(v) = \varphi(g).$$
□

Similar to Example 3.4, the assertions of Proposition 4.6 break down without convex-rangedness.

Example 4.7. Let $\Omega = \{1, 2, 3\}$ endowed with its power set Σ and define $\varphi\colon \mathcal{X} = \mathcal{X}_0 \longrightarrow \mathbb{R}$ by
$$\varphi(f) := \max_{\omega \in \Omega} \omega f(\omega).$$
We first claim that $\mathfrak{Ref}(\varphi) \neq \varnothing$. Indeed, suppose $\mathfrak{p} := \frac{1}{4}\delta_1 + \frac{1}{3}\delta_2 + \frac{5}{12}\delta_3$ (δ_ω denoting the point mass at $\omega \in \Omega$), then $f \sim_\mathfrak{p} g$ for $f, g \in \mathcal{X} = \mathcal{X}_0$ if and only if $f = g$. Hence, $\mathfrak{p} \in \mathfrak{Ref}(\varphi)$.
Next, we prove that all reference measures $\mathfrak{q} \in \mathfrak{Ref}(\varphi)$ are equivalent tp $\mathfrak{p}$. To this end, suppose $\mathfrak{q}(\{\sigma\}) = 0$ for some $\sigma \in \Omega$. Note that $f \sim_\mathfrak{q} g_n := f\mathbf{1}_{\Omega\setminus\{\sigma\}} + n\mathbf{1}_{\{\sigma\}}$ for all $(f, n) \in \mathcal{X} \times \mathbb{N}$, but $\varphi(g_n) \neq \varphi(g_k)$ for n, k large enough.
Third, $\mathfrak{Ref}(\varphi)$ does not contain the full equivalence class of $\mathfrak{p}$. Indeed, let $\mathfrak{q} := \frac{1}{3}(\delta_1 + \delta_2 + \delta_3)$ and note that $h := \mathbf{1}_{\{1\}} \sim_\mathfrak{q} \mathbf{1}_{\{3\}} =: g$ while $\varphi(h) = 1 \neq 3 = \varphi(g)$.

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let $A_\star \in \Sigma$ be a fixed event satisfying $0 < \mathfrak{p}(A_\star) < 1$ and assume that $\mathfrak{p} \neq \mathfrak{q}$. Define $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ by
$$T(u, v) := \varphi(u\mathbf{1}_{A_\star} + v\mathbf{1}_{A_\star^c}). \tag{4.2}$$

Proposition 4.6 yields for all $B \in \Sigma$ with $\mathfrak{p}(B) \in]0,1[$ that

$$T(u,v) = \varphi(u\mathbf{1}_B + v\mathbf{1}_{B^c}). \tag{4.3}$$

We shall prove that $\varphi(f) = T(m(f), M(f))$ for all $f \in \mathcal{X}_0$. Note that $f \sim_{\mathfrak{p}} g := f\mathbf{1}_{\{m(f)\le f\le M(f)\}} + m(f)\mathbf{1}_{\{f\notin]m(f),M(f)[\}}$ and that

$$m(f)\mathbf{1}_{\{g<M(f)\}} + M(f)\mathbf{1}_{\{g=M(f)\}} \le g \le m(f)\mathbf{1}_{\{g=m(f)\}} + M(f)\mathbf{1}_{\{g>m(f)\}}.$$

If there is no constant $c \in \mathbb{R}$ such that $f \sim_{\mathfrak{p}} c$, $\mathfrak{p}(g < M(f))$ and $\mathfrak{p}(g = m(f))$ lie in $]0,1[$. By (4.3),

$$\begin{aligned} T(m(f), M(f)) &= \varphi\left(m(f)\mathbf{1}_{\{g<M(f)\}} + M(f)\mathbf{1}_{\{g=M(f)\}}\right) \le \varphi(g) \\ &= \varphi(f) \le \varphi\left(m(f)\mathbf{1}_{\{g=m(f)\}} + M(f)\mathbf{1}_{\{g>m(f)\}}\right) = T(m(f), M(f)). \end{aligned}$$

□

5. Variants and extensions of Theorem 4.1

This section is devoted to extending the dichotomy in Theorem 4.1 under various assumptions.

5.1. Functionals defined on $\mathcal{X}$.

We first present a direct extension of Theorem 4.1 to functionals defined on *all* of $\mathcal{X}$. These additionally have to be norm-l.s.c.

Corollary 5.1. *Let $\mathfrak{p} \approx \mathfrak{q}$ be two convex-ranged probability charges on Σ. Suppose $\varphi\colon \mathcal{X} \longrightarrow \mathbb{R}$ is monotone, l.s.c., and satisfies $\{\mathfrak{p}, \mathfrak{q}\} \subset \mathfrak{Ref}(\varphi)$. Then one of the following alternatives holds:*

(1) $\mathfrak{p} = \mathfrak{q}$.
(2) $\varphi = T \circ (m, M)$ *for a function* $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$.

Proof. We first observe that the functionals m, M are monotone and translation invariant in that, for all $(f,r) \in \mathcal{X} \times \mathbb{R}$,

$$m(f+r) = m(f) + r \quad \text{and} \quad M(f+r) = M(f) + r.$$

A standard argument implies continuity of these functionals with respect to $\|\cdot\|_\infty$. Now let us assume that $\mathfrak{p} \neq \mathfrak{q}$. Defining T as in (4.2) and using Theorem 4.1, $\varphi(f) = T(m(f), M(f))$ holds for all $f \in \mathcal{X}_0$. For $f \in \mathcal{X}$ we set

$$f_n := \sum_{k\in\mathbb{Z}} k2^{-n}\mathbf{1}_{\{k2^{-n}\le f<(k+1)2^{-n}\}}, \quad n \in \mathbb{N}. \tag{5.1}$$

It can be observed that $(f_n)_{n\in\mathbb{N}} \subset \mathcal{X}_0$ and that $f_n \le f \le f_n + 2^{-n}$, $n \in \mathbb{N}$. Thus, monotonicity and lower semicontinuity imply

$$\varphi(f) = \sup_{n\in\mathbb{N}} \varphi(f_n) = \sup_{n\in\mathbb{N}} T(m(f_n), M(f_n)) = \sup_{n\in\mathbb{N}} \varphi\big(m(f_n)\mathbf{1}_{A_\star} + M(f_n)\mathbf{1}_{A_\star^c}\big),$$

$A_\star \in \Sigma$ being the event used in the definition of T. By continuity of m and M,

$$\lim_{n\to\infty} m(f_n)\mathbf{1}_{A_\star} + M(f_n)\mathbf{1}_{A_\star^c} = m(f)\mathbf{1}_{A_\star} + M(f)\mathbf{1}_{A_\star^c}.$$

This limit is furthermore monotone. Using monotonicity and lower semicontinuity of φ once more,

$$\sup_{n\in\mathbb{N}} \varphi\big(m(f_n)\mathbf{1}_{A_\star} + M(f_n)\mathbf{1}_{A_\star^c}\big) = \varphi\big(m(f)\mathbf{1}_{A_\star} + M(f)\mathbf{1}_{A_\star^c}\big) = T(m(f), M(f)).$$

□

We emphasise that φ in Corollary 5.1 has an extra property in comparison to Theorem 4.1, lower semicontinuity. This cannot be dropped, as illustrated by the following example.

Example 5.2. Define $\varphi\colon \mathcal{X} \longrightarrow \mathbb{R}$ by

$$\varphi(f) := \begin{cases} 0 & M(f) < 1, \\ 1 & M(f) = 1 \text{ and } \mathfrak{p}\,(f = M(f)) = 0, \\ 2 & M(f) > 1 \text{ or } M(f) = 1 \text{ and } \mathfrak{p}\,(f = M(f)) > 0. \end{cases}$$

By construction, φ is monotone and invariant under any probability charge $\mathfrak{q} \approx \mathfrak{p}$. However, φ fails alternative (2) in Corollary 5.1. Indeed, let $(A_n)_{n\in\mathbb{N}} \subset \Sigma$ be a sequence of events decreasing to $\varnothing$ such that $0 < \mathfrak{p}(A_n) < 1$ holds for all $n \in \mathbb{N}$. Moreover, set

$$f := \sum_{n=1}^{\infty} 2^{-n}\mathbf{1}_{A_n} \quad \text{and} \quad g = \mathbf{1}_{A_1}.$$

One readily verifies that $m(f) = m(g) = 0$, $M(f) = M(g) = 1$, and $\mathfrak{p}(g = 1) = \mathfrak{p}(A_1) > 0 = \mathfrak{p}(f = 1)$. Hence,

$$\varphi(f) = 1 < 2 = \varphi(g).$$

5.2. **Fatou and Lebesgue property.** The shape of φ is even more specific if norm lower semicontinuity is strengthened to the *Fatou property*. Like the Lebesgue property below, this is usually formulated in the presence of an underlying countably additive probability measure. We shall refrain from this assumption because the proof will work without. A functional $\varphi\colon \mathcal{X} \longrightarrow \mathbb{R}$ has the Fatou property if, for all norm-bounded sequences $(f_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ converging pointwise to some limit $f \in \mathcal{X}$, the estimate $\varphi(f) \le \liminf_{n\to\infty} \varphi(f_n)$ holds.

Corollary 5.3. *Let $\mathfrak{p} \approx \mathfrak{q}$ be two convex-ranged probability charges on Σ. Suppose $\varphi\colon \mathcal{X} \longrightarrow \mathbb{R}$ is monotone, has the Fatou property, and satisfies $\{\mathfrak{p}, \mathfrak{q}\} \subset \mathfrak{Ref}(\varphi)$. Then one of the following alternatives holds:*

(1) $\mathfrak{p} = \mathfrak{q}$.
(2) $\varphi = \varphi \circ M$.

Proof. Suppose $\mathfrak{p} \neq \mathfrak{q}$ and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of events decreasing to $\varnothing$ satisfying $\mathfrak{p}(A_n) = \frac{1}{n}$, $n \in \mathbb{N}$. The existence of such a sequence is guaranteed by convex-rangedness of $\mathfrak{p}$. Let $f \in \mathcal{X}$ be arbitrary. By Corollary 5.1 and the Fatou property,

$$\varphi(f) = \liminf_{n\to\infty} \varphi\left(m(f)\mathbf{1}_{A_n} + M(f)\mathbf{1}_{A_n^c}\right) \ge \varphi(M(f)). \tag{5.2}$$

Moreover, for $f \in \mathcal{X}_0$, $f \sim_{\mathfrak{p}} f \wedge M(f)$ holds, which in conjunction with monotonicity of φ yields

$$\varphi(f) = \varphi(f \wedge M(f)) \leq \varphi(M(f)). \tag{5.3}$$

Combining (5.2) and (5.3) for simple functions, one obtains $\varphi|_{\mathcal{X}_0} = (\varphi \circ M)|_{\mathcal{X}_0}$. For arbitrary $f \in \mathcal{X}$ define a sequence $(f_n)_{n\in\mathbb{N}}$ of simple random variables as in (5.1). Because of monotonicity and the Fatou property,

$$\varphi(f) = \sup_{n\in\mathbb{N}} \varphi(f_n) = \sup_{n\in\mathbb{N}} \varphi\big(M(f_n)\big) \leq \varphi\big(M(f)\big).$$

Combining the latter estimate with (5.2), we conclude the proof. □

In the following proposition we assume the even stronger *Lebesgue property*, i.e., both φ and $-\varphi$ have the Fatou property, and can thereby drop monotonicity of φ from the list of assumptions. Again, if φ is additionally convex, the Lebesgue property can typically not hold without countable additivity of all reference probabilities. As the proof does not require this property, we shall abstain from it.

Proposition 5.4. *Let $\mathfrak{p} \approx \mathfrak{q}$ be two convex-ranged probability charges on Σ. Suppose $\varphi\colon \mathcal{X} \longrightarrow \mathbb{R}$ has the Lebesgue property and satisfies $\{\mathfrak{p}, \mathfrak{q}\} \subset \mathfrak{Ref}(\varphi)$. Then $\mathfrak{p} = \mathfrak{q}$ or φ is constant.*

Proof. Let $f \in \mathcal{X}_0$ be a simple function and fix a value $y \in \mathcal{W}_{\mathfrak{p}}(f)$. Select a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{X}_0$ such that $\mathcal{W}_{\mathfrak{p}}(f) = \mathcal{W}_{\mathfrak{p}}(f_n)$, $n \in \mathbb{N}$, and such that $\{f_n = y\} \uparrow \Omega$ as $n \to \infty$. By Proposition 4.6, $\varphi(f) = \varphi(f_n)$. As $f_n \to y$ pointwise as $n \to \infty$, the Lebesgue property implies

$$\varphi(f) = \lim_{n\to\infty} \varphi(f_n) = \varphi(y).$$

This is sufficient to prove constancy of φ on $\mathcal{X}_0$. For arbitrary $f \in \mathcal{X}$ define a sequence $(f_n)_{n\in\mathbb{N}}$ of simple random variables as in (5.1). This sequence is norm bounded and converges to f pointwise. Thus, $\varphi(f) = \lim_{n\to\infty} \varphi(f_n) = \varphi(0)$. □

5.3. **Dilatation monotonicity.** Another situation in which we manage to dispense with the assumption of monotonicity is studied in Theorem 5.8 below. As preparation, we demonstrate that the assumption of monotonicity in Corollary 5.1 cannot be dropped.

Example 5.5. The 2-Lipschitz continuous map $\varphi\colon \mathcal{X}_0 \longrightarrow \mathbb{R}$ constructed in Example 4.2 uniquely extends to $\mathcal{X}$ by setting $\varphi(f) := \lim_{n\to\infty} \varphi(f_n)$ whenever $(f_n)_{n\in\mathbb{N}} \subset \mathcal{X}_0$ converges to f in norm. Fix a probability charge $\mathfrak{q} \approx \mathfrak{p}$ and suppose $f, g \in \mathcal{X}$ satisfy $f \sim_{\mathfrak{q}} g$. Then the sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ constructed in (5.1) satisfy $f_n \sim_{\mathfrak{q}} g_n$, $n \in \mathbb{N}$. In particular, $\mathcal{W}_{\mathfrak{p}}(f_n) = \mathcal{W}_{\mathfrak{p}}(g_n)$, whence $\varphi(f_n) = \varphi(g_n)$ follows for all $n \in \mathbb{N}$ from the observations in Example 4.2. We conclude from the definition of φ that $\varphi(f) = \varphi(g)$. In total, φ is 2-Lipschitz continuous, fails alternative (2) in Corollary 5.1, and satisfies $\mathfrak{Ref}(\varphi) = \{\mathfrak{q} \mid \mathfrak{q} \approx \mathfrak{p}\}$.

Theorem 5.8 proves the dichotomy in Corollary 5.1 for functionals which are *not* monotone, but have the property of *dilatation monotonicity*. The latter is well known if the reference measure $\mathbb{P}$ is countably additive and usually implies law invariance; see, e.g., [6, 22] and the references therein. We shall stick to the guiding thread of not imposing countable additivity of the reference probability and therefore first extend conditional expectations to the case of probability charges $\mathfrak{p}$. Let Π be the set of all finite measurable partitions of Ω, and set

$$\mathbb{E}_{\mathfrak{p}}[f|\pi] := \sum_{B\in\pi:\ \mathfrak{p}(B)>0} \frac{\int f\mathbf{1}_B \,\mathrm{d}\mathfrak{p}}{\mathfrak{p}(B)}\mathbf{1}_B \in \mathcal{X}_0, \quad (f,\pi)\in\mathcal{X}\times\Pi.$$

If $\mathfrak{p}=\mathbb{P}$ for a countably additive probability measure, this is precisely a version of the conditional expectation of f under $\mathbb{P}$ with respect to the σ-algebra $\sigma(\pi)$ generated by π.

Definition 5.6. Let $\mathcal{X}_0\subset\mathcal{D}\subset\mathcal{X}$ be a set. A functional $\varphi\colon\mathcal{D}\longrightarrow\mathbb{R}$ is $\mathfrak{p}$-*dilatation monotone* if, for all $f\in\mathcal{D}$ and all $\pi\in\Pi$,

$$\varphi\left(\mathbb{E}_{\mathfrak{p}}[f|\pi]\right)\leq\varphi(f). \tag{5.4}$$

Remark 5.7.

(1) Given a countably additive reference measure, dilatation monotonicity is often defined by the demand that inequality (5.4) holds for arbitrary conditional expectations. As that has no immediate counterpart under a finitely additive reference measure, Definition 5.6 instead departs from the comparatively most general notion of dilatation monotonicity employed, for instance, in [17, 22].

(2) If $\varphi\colon\mathcal{X}\longrightarrow\mathbb{R}$ is l.s.c. and $\mathbb{P}$-dilatation monotone for a countably additive and atomless probability measure $\mathbb{P}$, then $\mathbb{P}\in\mathfrak{Ref}(\varphi)$. The earliest formulation of this observation is due to [6] and has since been extended to various model spaces and weaker notions of dilatation monotonicity such as the one given above.

We are now ready to formulate our main result concerning dilatation-monotone functionals on the space $\mathcal{X}_0$ of simple functions.

Theorem 5.8. *Let $\mathfrak{p}\approx\mathfrak{q}$ be two convex-ranged probability charges on Σ. Suppose $\varphi\colon\mathcal{X}\longrightarrow\mathbb{R}$ is l.s.c., satisfies $\{\mathfrak{p},\mathfrak{q}\}\subset\mathfrak{Ref}(\varphi)$, and is $\mathfrak{p}$-dilatation monotone. Then one of the following alternatives holds:*

(1) $\mathfrak{p}=\mathfrak{q}$.
(2) $\varphi = T\circ(m,M)$ *for a function* $T\colon\mathbb{R}^2\longrightarrow\mathbb{R}$ *satisfying* $T(x,y)\leq T(x',y')$ *whenever* $x'\leq x\leq y\leq y'$.

In that case, φ is dilatation monotone with respect to every convex-ranged probability measure $\mathfrak{r}\approx\mathfrak{p}$.

Proof. As usual, suppose $\mathfrak{p}\neq\mathfrak{q}$ and define the function $T\colon\mathbb{R}^2\longrightarrow\mathbb{R}$ like in (4.2). We prove first that, for all $f\in\mathcal{X}_0$, $\varphi(f)=T\left(m(f),M(f)\right)$. This identity clearly holds if $f\sim_{\mathfrak{p}} c$ for some constant $c\in\mathbb{R}$. We can thus assume that $\mathcal{W}_{\mathfrak{p}}(f)$ contains at least two values. We proceed by induction over $n:=|\mathcal{W}_{\mathfrak{p}}(f)|$.
<u>$n=2$:</u> If $n=2$, this follows from Proposition 4.6.
<u>$n\to n+1$:</u> Suppose $m(f)=x_1<x_2<\ldots<x_{n+1}=M(f)$ are the elements in $\mathcal{W}_{\mathfrak{p}}(f)$. Moreover, we can assume without loss that $\{f\notin\mathcal{W}_{\mathfrak{p}}(f)\}=\varnothing$. Select a sequence of events $(A_k)_{k\in\mathbb{N}}\subset\Sigma$ decreasing to $\varnothing$ such that $A_k\subset\{f=x_1 \text{ or } f=x_2\}$ and $\mathfrak{p}(A_k)=\frac{1}{k}\mathfrak{p}(f=$

x_1 or $f = x_2$), $k \in \mathbb{N}$. For all $k \in \mathbb{N}$, we furthermore define

$$f_k := f\mathbf{1}_{\{f \notin \{x_1,x_2\}\}} + x_2\mathbf{1}_{A_k} + x_1\mathbf{1}_{\{f\in\{x_1,x_2\}\}\cap A_k^c}.$$

For all $k \in \mathbb{N}$, $\mathcal{W}_{\mathfrak{p}}(f) = \mathcal{W}_{\mathfrak{p}}(f_k)$, whence we conclude with Proposition 4.6 that $\varphi(f) = \liminf_{k\to\infty}\varphi(f_k)$. Setting

$$\pi := \big\{\{f \le x_2\}, \{f = x_3\}, \dots, \{f = x_{n+1}\}\big\},$$

$\mathfrak{p}$-dilatation monotonicity, lower semicontinuity, and the induction hypothesis imply

$$\begin{aligned}\varphi(f) &= \liminf_{k\to\infty}\varphi(f_k) \ge \liminf_{k\to\infty}\varphi\left(\mathbb{E}_{\mathfrak{p}}[f_k|\pi]\right)\\ &\ge \varphi\Big(x_1\mathbf{1}_{\{f\in\{x_1,x_2\}\}} + \sum_{i=3}^{n+1} x_i\mathbf{1}_{\{f=x_i\}}\Big) = T(x_1, x_{n+1}).\end{aligned}$$

Next, let $B_1, \dots, B_n \in \Sigma$ be events partitioning Ω and satisfying $\mathfrak{p}(B_i) = \frac{1}{n}$, $1 \le i \le n$. Let $\lambda \in]0,1[$ be such that $x_2 = \lambda x_1 + (1-\lambda)x_3$. Select events $A_i \subset B_i$, $i = 1, 2$, such that $\mathfrak{p}(A_1) = \frac{\lambda}{n}$ and $\mathfrak{p}(A_2) = \frac{1-\lambda}{n}$. Consider the partition $\widehat{\pi} := \{B_1\setminus A_1, A_1\cup A_2, B_2\setminus A_2, B_3, \dots, B_n\}$ and observe that

$$\mathbb{E}_{\mathfrak{p}}\Big[x_1\mathbf{1}_{B_1} + \sum_{i=2}^{n} x_{i+1}\mathbf{1}_{B_i}\Big|\mathcal{H}\Big] = x_1\mathbf{1}_{B_1\setminus A_1} + x_2\mathbf{1}_{A_1\cup A_2} + x_3\mathbf{1}_{B_2\setminus A_2} + \sum_{i=3}^{n} x_{i+1}\mathbf{1}_{B_i} =: g.$$

Using the induction hypothesis, Proposition 4.6, and dilatation monotonicity,

$$T(x_1, x_{n+1}) = \varphi\Big(x_1\mathbf{1}_{B_1} + \sum_{i=2}^{n} x_{i+1}\mathbf{1}_{B_i}\Big) \ge \varphi(g) = \varphi(f).$$

In total, $\varphi(f) = T(m(f), M(f))$. The induction is complete.

Now we extend the identity $\varphi = T\circ(m, M)$ to all of $\mathcal{X}$. Fix $f \in \mathcal{X}$. If $M(f) = m(f)$, set $\delta = \delta(f) := 1$, else we set it to be $M(f) - m(f)$. For given $n \in \mathbb{N}$, there are at most finitely many events

$$A_k^n := \{m(f) + (k-1)2^{-n}\delta \le f < m(f) + k2^{-n}\delta\}, \quad k \in \mathbb{Z},$$

that are nonempty because f is bounded. This allows us to define functions $f_n^{\pm} \in \mathcal{X}_0$ by

$$f_n^- := \sum_{k\in\mathbb{Z}}(m(f) + (k-1)2^{-n}\delta)\mathbf{1}_{A_k^n} \quad\text{and}\quad f_n^+ := \sum_{k\in\mathbb{Z}}(m(f) + k2^{-n}\delta)\mathbf{1}_{A_k^n}.$$

Moreover, set $g_n \in \mathcal{X}_0$ to be

$$g_n(\omega) := \begin{cases} f_n^+(\omega) & \text{if } f_n^+(\omega) \le m(f) \text{ or } f_n^+(\omega) = M(f)\\ f_n^-(\omega) & \text{else.}\end{cases}$$

By construction, $\|f - g_n\|_\infty \le 2^{-n}\delta$ for all $n \in \mathbb{N}$. We furthermore claim that, for all $n \in \mathbb{N}$,

$$m(g_n) = m(f) \quad\text{and}\quad M(g_n) = M(f).$$

We verify the latter identity only, the former can be shown analogously. Let $\ell \in \mathbb{Z}$ such that $m(f) + \ell 2^{-n}\delta = M(f)$. By definition, $g_n = M(f)$ on $A_\ell^n \cup A_{\ell+1}^n$. Moreover, for $s, t \in \mathbb{R}$

satisfying $M(f) - 2^{-n}\delta < s < M(f) < t < M(f) + 2^{-n}\delta$,

$$\mathfrak{p}(A^n_\ell \cup A^n_{\ell+1}) \geq \mathfrak{p}(s < f \leq t) = \mathfrak{p}(f \leq t) - \mathfrak{p}(f \leq s) > 0.$$

Hence, $\mathfrak{p}(g_n = M(f)) > 0$ and $M(g_n) \geq M(f)$. On the other hand, $M(g_n) > M(f)$ can only hold if $\mathfrak{p}(A^n_{\ell+r}) > 0$ for some $r \geq 2$. However, choosing t as before, $\mathfrak{p}(A^n_{\ell+r}) \leq (f > t) = 0$, which means that $M(g_n) = M(f)$.

Now, lower semicontinuity of φ yields that

$$\varphi(f) \leq \liminf_{n\to\infty} \varphi(g_n) = T\left(m(g_n), M(g_n)\right) = T\left(m(f), M(f)\right).$$

The identity $\varphi = T\circ(m, M)$ is proved if we can additionally demonstrate the converse estimate. To this effect, let $f \in \mathcal{X}$ be arbitrary, set $B^n_k := \{f \in [k2^{-n}, (k+1)2^{-n})\}$, $(k, n) \in \mathbb{Z} \times \mathbb{N}$, and define

$$\pi_n := \{B^n_k \mid k \in \mathbb{Z},\, B^n_k \neq \varnothing\}, \quad n \in \mathbb{N}.$$

For all $n \in \mathbb{N}$ and all $k \in \mathbb{Z}$ such that $\mathfrak{p}(B^n_k) > 0$, we have $f^-_n \leq \mathbb{E}_\mathfrak{p}[f|\pi_n] \leq f^+_n$ on B^n_k. Consequently, $m(f^-_n) \leq m\big(\mathbb{E}_\mathfrak{p}[f|\pi_n]\big) \leq m(f^+_n) = m(f^-_n) + 2^{-n}$ and $M(f^-_n) \leq M\big(\mathbb{E}_\mathfrak{p}[f|\pi_n]\big) \leq M(f^+_n) = M(f^-_n)+2^{-n}$, and we obtain $\lim_{n\to\infty} M\left(\mathbb{E}_\mathfrak{p}[f|\pi_n]\right) = M(f)$ and $\lim_{n\to\infty} m\left(\mathbb{E}_\mathfrak{p}[f|\pi_n]\right) = m(f)$. Thus, for arbitrary $A \in \Sigma$ with $\mathfrak{p}(A) \in]0, 1[$,

$$\begin{aligned}
\varphi(f) &\geq \limsup_{n\to\infty} \varphi\left(\mathbb{E}_\mathfrak{p}[f|\pi_n]\right) \geq \liminf_{n\to\infty} \varphi\left(\mathbb{E}_\mathfrak{p}[f|\pi_n]\right)\\
&= \liminf_{n\to\infty} \varphi\big(M\left(\mathbb{E}_\mathfrak{p}[f|\pi_n]\right)\mathbf{1}_A + m\left(\mathbb{E}_\mathfrak{p}[f|\pi_n]\right)\mathbf{1}_{A^c}\big)\\
&\geq \varphi\big(M(f)\mathbf{1}_A + m(f)\mathbf{1}_{A^c}\big) = T(M(f), m(f)).
\end{aligned}$$

We now prove that, for all $x' \leq x \leq y \leq y'$ we have $T(x, y) \leq T(x', y')$. This is clear if $x' = y'$ and we can turn immediately to the case $x' < y'$. Even more, we shall assume $x' < x < y < y'$. Let $\alpha, \beta \in (0, 1)$ be such that

$$\alpha x' + (1 - \alpha)y' = x \quad \text{and} \quad \beta x' + (1 - \beta)y' = y.$$

Partition Ω into events $A_1, ..., A_4 \in \Sigma$ with positive $\mathfrak{p}$-probability such that $\mathfrak{p}(A_1) = \frac{\alpha}{2}$, $\mathfrak{p}(A_3) = \frac{\beta}{2}$, and $\mathfrak{p}(A_1 \cup A_2) = \mathfrak{p}(A_3 \cup A_4) = \frac{1}{2}$. The random variable

$$f := x'\mathbf{1}_{A_1\cup A_3} + y'\mathbf{1}_{A_2\cup A_4}$$

satisfies $\varphi(f) = T(x', y')$ by construction. For the partition $\pi := \{A_1 \cup A_2, A_3 \cup A_4\} \in \Pi$ we compute

$$\mathbb{E}_\mathfrak{p}[f|\pi] = x\mathbf{1}_{A_1\cup A_2} + y\mathbf{1}_{A_3\cup A_4},$$

whence we conclude with dilatation monotonicity that

$$T(x, y) = \varphi\big(\mathbb{E}_\mathfrak{p}[f|\pi]\big) \leq \varphi(f) = T(x', y').$$

Now, for $x' \leq x \leq y \leq y'$ we may select sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converging to x and y, respectively, such that, for all $n \in \mathbb{N}$, $x' < x_n < y_n < y'$. As T is l.s.c. by construction,

$$T(x, y) \leq \liminf_{n\to\infty} T(x_n, y_n) \leq T(x', y').$$

The last claim requires to establish that, for every convex-ranged probability charge $\mathfrak{q} \approx \mathfrak{p}$, φ is $\mathfrak{q}$-dilatation monotone. Observe that, for all $f \in \mathcal{X}$, $\pi \in \Pi$, $B \in \Pi$ with $\mathfrak{q}(B) > 0$, $s < m(f)$, and $t > M(f)$, we have

$$s \leq \tfrac{1}{\mathfrak{q}(B)} \int f \mathbf{1}_B \, \mathrm{d}\mathfrak{q} \leq t.$$

Hence, for all $\pi \in \Pi$, $m(f) \leq m(\mathbb{E}_{\mathfrak{q}}[f|\pi])$ and $M(\mathbb{E}_{\mathfrak{q}}[f|\pi]) \leq M(f)$. Given what we have proved about T in assertion (2),

$$\varphi\big(\mathbb{E}_{\mathfrak{q}}[f|\pi]\big) = T\big(m(\mathbb{E}_{\mathfrak{q}}[f|\pi]), M(\mathbb{E}_{\mathfrak{q}}[f|\pi])\big) \leq T(m(f), M(f)) = \varphi(f).$$

□

Theorem 5.8 fails without the assumption of lower semicontinuity:

Example 5.9. Let $\mathfrak{p}$ be a convex-ranged probability charge on (Ω, Σ) and define $\varphi\colon \mathcal{X} \longrightarrow \{0, 1\}$ by

$$\varphi(f) = \begin{cases} 0 & \text{if } f \sim_{\mathfrak{p}} g \text{ for some } g \in \mathcal{X}_0, \\ 1 & \text{else.} \end{cases}$$

Clearly, φ is not l.s.c., but by construction dilatation monotone with respect to and invariant under every convex-ranged $\mathfrak{q} \approx \mathfrak{p}$.

An important field of application of Theorem 5.8 are *quasiconvex* l.s.c. functionals $\varphi\colon \mathcal{X} \longrightarrow \mathbb{R}$, i.e., their lower level sets $\{f \in \mathcal{X} \mid \varphi(f) \leq c\}$, $c \in \mathbb{R}$, are not only closed, but also convex. Up to a change of sign, they can also be seen as numerical representations of (not necessarily monotone) continuous and convex preferences over $\mathcal{X}$ as used, for instance, in [17]. If for such a functional we find a countably additive and atomless $\mathbb{P} \in \mathfrak{Ref}(\varphi)$, then φ is also $\mathbb{P}$-dilatation monotone (as can be seen using [26, Lemma 1.3]). Theorem 5.8 then proves that $\mathfrak{Ref}(\varphi) = \{\mathbb{P}\}$ unless φ is of the specific shape described in alternative (2). A prominent subclass of the previous family of functionals are *convex monetary risk measures*; cf. [14, Chapter 4].
We close this section with a detailed comparison of the present results to the already mentioned paper [7].

Remark 5.10. Given a domain of definition $\mathcal{D}$, a functional $\varphi\colon \mathcal{D} \to \mathbb{R}$ induces a *preference relation* $\preceq_\varphi$ over the elements of $\mathcal{D}$ by setting $f \preceq_\varphi g$ ("g is weakly preferred to f") if $\varphi(f) \leq \varphi(g)$. Relation $\preceq_\varphi$ is nontrivial if and only if φ is nonconstant. More generally, a preference relation $\preceq$ is $\mathfrak{p}$-invariant if $f \sim_{\mathfrak{p}} g$ implies indifference $f \sim g$, i.e., $f \preceq g$ and $g \preceq f$. The operation of switching from φ to $\preceq_\varphi$ puts the results in the present paper in proximity of an earlier contribution by Chew & Sagi [7] axiomatising probabilistically sophisticated preference relations.
Unfolding the setting of [7] in our framework, one starts with a nontrivial, complete, and transitive binary relation $\preceq$ on $\mathcal{X}_0$.[3] Based on these preferences, two events $A, B \in \Sigma$ are declared exchangeable ($A \approx B$) if, for all $x, y \in \mathbb{R}$ and $f \in \mathcal{X}_0$,

$$x\mathbf{1}_A + y\mathbf{1}_B + f\mathbf{1}_{(A\cup B)^c} \sim y\mathbf{1}_A + x\mathbf{1}_B + f\mathbf{1}_{(A\cup B)^c}.$$

[3] Note that [7] considers events from an underlying algebra, not a σ-algebra as in the present paper.

Exchangeability leads to a comparative likelihood relation $\trianglelefteq$ on Σ by setting $A \trianglelefteq B$ whenever $B \setminus A$ contains a subevent C such that $A \setminus B \approx C$.

The goal of [7] is to establish probabilistic sophistication, i.e., the existence of a *unique* agreeing finitely additive probability $\mathfrak{p}$ on Σ, satisfying $A \trianglelefteq B$ if and only if $\mathfrak{p}(A) \leq \mathfrak{p}(B)$, $A, B \in \Sigma$. This is achieved by means of the following three axioms:

Axiom 1: Every sequence $(A_n)_{n\in\mathbb{N}} \subset \Sigma$ consisting of pairwise disjoint events such that $A_n \approx A_{n+1}$, $n \in \mathbb{N}$, must begin with a null event A_1.[4]

Axiom 2: $\trianglelefteq$ is complete: For all $A, B \in \Sigma$, $A \trianglelefteq B$ or $B \trianglelefteq A$.

Axiom 3: Suppose $A, B, C \in \Sigma$ are pairwise disjoint such that $A \approx B$ and C is not null. Then $B \triangleleft A \cup C$.

Their main result [7, Theorem 1] then reads as follows:

> *The comparative likelihood relation $\trianglelefteq$ satisfies Axioms 1–3 if and only if there is a unique and* solvable *probability charge $\mathfrak{p}$ agreeing with $\trianglelefteq$.*[5] *If $\mathfrak{p}$ has no atoms, then $A \approx B$ holds if and only if $\mathfrak{p}(A) = \mathfrak{p}(B)$, and if $f \sim_{\mathfrak{p}} g$, then $f \sim g$.*

Consequently, if the preference relation $\preceq$ has a numerical representation $\varphi\colon \mathcal{X} \longrightarrow \mathbb{R}$, i.e., $\preceq = \preceq_\varphi$, and if the agreeing probability charge $\mathfrak{p}$ has no atoms, then $\mathfrak{p} \in \mathfrak{Ref}(\varphi)$.

In view of the comparison to functionals above, one may be tempted to think of the uniqueness results presented in this paper as "functional counterparts" of the Chew-Sagi Theorem. Both contributions share an approach of maximal generality; while [7] imposes effectively no conditions on the preferences it applies to, we formulate minimal conditions under which our results hold. However, the present results do not follow from the Chew-Sagi Theorem due to the following key problem.

If a functional $\varphi\colon \mathcal{X}_0 \longrightarrow \mathbb{R}$ is $\mathfrak{p}$-invariant and two events $A, B \in \Sigma$ satisfy $\mathfrak{p}(A) = \mathfrak{p}(B)$, then A and B are exchangeble: $A \approx B$. If $\mathfrak{p}$ is additionally convex-ranged, $\mathfrak{p}(A) \leq \mathfrak{p}(B)$ implies $A \trianglelefteq B$. However, it is not clear whether converse implications hold, i.e., whether $A \approx B$ implies $\mathfrak{p}(A) = \mathfrak{p}(B)$. Without properly understanding the exchangeability relation $\approx$, it should be very challenging to verify that Axioms 1–3 hold for the preference relation $\preceq = \preceq_\varphi$ though. Chew & Sagi also acknowledge the limitations of an exchangeability-based approach to probabilistic sophistication in their discussion surrounding [7, Example 2]. Building on the latter, consider an atomless probability measure $\mathbb{P}$ on (Ω, Σ) as well as the variance functional $\varphi\colon \mathcal{X}_0 \to \mathbb{R}$ defined by $\varphi(f) = \mathbb{E}_{\mathbb{P}}[f^2] - \mathbb{E}_{\mathbb{P}}[f]^2$. For all $A \in \mathcal{F}$ with $\mathbb{P}(A) \in [0, \frac{1}{2}]$, $A \approx A^c$ holds, so [7, Theorem 1] fails. However, applying Propositions 3.1 and 4.6, the uniqueness statement $\mathfrak{Ref}(\varphi) = \{\mathbb{P}\}$ follows (even though this may admittedly look like using a sledgehammer to crack a nut).

In view of the preceding point, it might also *a priori* be difficult to decide if the existence of more than one convex-ranged reference probability implies a failure of Axioms 1–3. By Proposition 4.6, we see for a convex-ranged $\mathfrak{p} \in \mathfrak{Ref}(\varphi)$ and all $f, g \in \mathcal{X}_0$ that $\mathcal{W}_{\mathfrak{p}}(f) = \mathcal{W}_{\mathfrak{p}}(g)$

[4] An event $N \in \Sigma$ is null if, for all $f, g, h \in \mathcal{X}_0$, $f\mathbf{1}_N + h\mathbf{1}_{N^c} \sim g\mathbf{1}_N + h\mathbf{1}_{N^c}$.

[5] A probability charge $\mathfrak{p}$ is solvable if, for all $A, B \in \Sigma$ such that $\mathfrak{p}(A) \leq \mathfrak{p}(B)$, one finds a subevent C of B such that $\mathfrak{p}(C) = \mathfrak{p}(A)$.

implies $\varphi(f) = \varphi(g)$. Hence, all $A, B \in \Sigma$ with $0 < \mathfrak{p}(A), \mathfrak{p}(B) < 1$ satisfy $A \approx B$. No agreeing probability measure can exist, and Axioms 1 and 3 are violated.
From these points of view, the present paper can be viewed as a collection of "quantitative" Chew-Sagi type results. On another note, the results of [7] are tailored to preference relations whose numerical representability is irrelevant. In many other fields of applications such as risk measures, this perspective might not be ideal because the objects of primary interest are functionals on random variables. Besides, our proof technique is completely different from [7], and the extensions to the larger space $\mathcal{X}$ and finer breakdown of the results in Section 4 presented in Section 5 above — for instance, the observations concerning dilatation monotone functionals in Theorem 5.8 — have no counterpart in that paper.

6. Functionals defined on unbounded random variables

In this short section we generalise some of the previous results to real-valued functionals defined on spaces of *unbounded* random variables. Deviating from the setting in previous sections, we facilitate our considerations by always assuming that a given *countably additive* and atomless probability measure $\mathbb{P}$ on Σ is a reference probability. Moreover, we look at the spaces L^p of equivalence classes of random variables with finite p-th moment up to $\mathbb{P}$-a.s. equality; cf. [1, Chapter 13]. In contrast to their representatives, equivalence classes will be denoted by capital letters. These spaces are equipped with their canonical Lebesgue topologies if $p \in [1, \infty)$, with the metric $(X, Y) \mapsto \mathbb{E}_{\mathbb{P}}[|X - Y|^p]$ if $0 < p < 1$ (cf. [1, Theorem 13.30]), and by convergence in $\mathbb{P}$-probability if $p = 0$. Lower semicontinuity and continuity of functionals φ will be understood with respect to these — in comparison with the norm topology of $\mathcal{X}$ or L^∞ — weaker topologies. In turn, this also allows to say more about functionals lacking monotonicity, now understood with respect to the $\mathbb{P}$-*a.s. order*. If another probability measure $\mathbb{Q} \approx \mathbb{P}$ is given, then convergence in $\mathbb{P}$-probability is the same as convergence in $\mathbb{Q}$-probability, the topologies on L^p, $0 < p < \infty$, generally depend on $\mathbb{P}$ irreducibly.
We begin with a generalisation of Corollary 5.1.

Corollary 6.1. *Let $\mathfrak{q} \approx \mathbb{P}$ be another convex-ranged probability charge on Σ. Suppose $\{\mathbb{P}, \mathfrak{q}\} \subset \mathfrak{Ref}(\varphi)$ for a monotone and l.s.c. functional $\varphi \colon L^0 \longrightarrow \mathbb{R}$. Then either $\mathbb{P} = \mathfrak{q}$ or*

$$\varphi(X) = \begin{cases} \varphi\left(M(X)\right) & M(X) < \infty \\ \sup(\varphi|_{\mathbb{R}}) & M(X) = \infty, \end{cases} \qquad X \in L^0.$$

Proof. Suppose $\mathbb{P} \neq \mathfrak{q}$ and let $\iota \colon \mathcal{X} \longrightarrow L^\infty$ be the canonical map. Define $\widehat{\varphi} := \varphi \circ \iota$ on $\mathcal{X}$ and note that it has the Fatou property. By Corollary 5.3, $\widehat{\varphi} = \widehat{\varphi} \circ M$, whence we also conclude $\varphi = \varphi \circ M$. Next assume $X \in L^0$ is a discrete random variable bounded above. In particular, the set $\{x \in \mathbb{R} \mid \mathbb{P}(X = x) > 0\}$ is countable. Using Lemma 4.4 and adapting the argument underlying the proof of Proposition 4.6, we can construct a sequence $(X_n)_{n \in \mathbb{N}} \subset L^0$ such that $\mathbb{P}(X = x) > 0$ if and only if $\mathbb{P}(X_n = x) > 0$, $x \in \mathbb{R}$, $\varphi(X) = \varphi(X_n)$, and $\lim_{n\to\infty} \mathbb{P}(X_n =$

$M(X)) = 1$. By lower semicontinuity,

$$\varphi(M(X)) \leq \liminf_{n\to\infty} \varphi(X_n) = \varphi(X).$$

The inequality $\varphi(X) \leq \varphi(M(X))$ holds by monotonicity. Now let $X \in L^0$ be arbitrary. Set

$$X_n := \sum_{k=-\infty}^{n2^n} k2^{-n}\mathbf{1}_{\{X\in[k2^{-n},(k+1)2^{-n})\}}, \quad n \in \mathbb{N}.$$

Then $X_n \leq X$ $\mathbb{P}$-a.s. for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} X_n = X$ in L^0. If X is bounded above, then monotonicity and lower semicontinuity imply

$$\varphi(X) = \lim_{n\to\infty} \varphi(X_n) = \lim_{n\to\infty} \varphi(M(X_n)) = \varphi(M(X)).$$

If X is unbounded above, then monotonicity shows

$$\varphi(X) = \lim_{n\to\infty} \varphi(X_n) = \lim_{n\to\infty} \varphi(n) = \sup(\varphi|_{\mathbb{R}}).$$

□

The next result leverages Proposition 5.4.

Corollary 6.2. *Let $\mathfrak{q} \approx \mathbb{P}$ be another convex-ranged probability charge on Σ. Suppose $\{\mathbb{P}, \mathfrak{q}\} \subset \mathfrak{Ref}(\varphi)$ for a continuous functional $\varphi\colon L^p \longrightarrow \mathbb{R}$, $p \in [0, \infty)$. Then $\mathbb{P} = \mathfrak{q}$ or φ is constant.*

Proof. If $\mathbb{P} \neq \mathfrak{q}$, then the functional $\widehat{\varphi}\colon \mathcal{X} \longrightarrow \mathbb{R}$ defined as in the proof of Corollary 6.1 has the Lebesgue property and is therefore constant by Proposition 5.4. Next, note that for all $X \in L^p$ the sequence defined by $X_n := X\mathbf{1}_{\{|X|\leq n\}} \in L^\infty$ converges in L^p to X as $n \to \infty$. Thus

$$\varphi(0) = \lim_{n\to\infty} \varphi(X_n) = \varphi(X).$$

□

The proof of Corollary 6.2 only makes use of the inclusion $L^\infty \subset L^p$ and the so-called order continuity of the topology on L^p. The result therefore holds *verbatim* on more general order-continuous topological vector spaces $\mathcal{X} \subset L^0$ satisfying $L^\infty \subset \mathcal{X}$, for instance, all Orlicz hearts. Corollary 6.2 also fails without putting any assumption on φ. Indeed, consider $\varphi\colon L^p \longrightarrow \mathbb{R}$ defined by $\varphi(X) = 0$ if $X \in L^\infty$ and $\varphi(X) = 1$ otherwise. Then $\mathfrak{Ref}(\varphi) = \{\mathfrak{q} \mid \mathfrak{q} \approx \mathbb{P}\}$ and $\varphi|_{L^\infty}$ has the Lebesgue property, but φ is not constant.

7. Scenario-based functionals

In this final section, we go beyond $\mathfrak{p}$-invariance. This is directly motivated by the *scenario-based risk measures studied* in [27]. It could well be too stringent a demand that a functional φ values every pair of random variables $f \sim_{\mathfrak{p}} g$ equally. Instead, [27] suggests the more general notion of scenario-basedness that contains $\mathfrak{p}$-invariance as a special case.

Definition 7.1. Suppose $\mathfrak{Q}$ is a finite set of convex-ranged probability charges on Σ and $\mathcal{D} \subset \mathcal{X}$ is a domain of definition of a real-valued functional φ. We then call φ $\mathfrak{Q}$-*based* if, for all $f, g \in \mathcal{D}$, $\varphi(f) = \varphi(g)$ whenever $f \sim_{\mathfrak{q}} g$ holds for all $\mathfrak{q} \in \mathfrak{Q}$.

Clearly, φ is $\mathfrak{p}$-invariant if and only if it is $\{\mathfrak{p}\}$-based.

The present goal is to adapt the technique underlying Lemma 4.4 and Proposition 4.6 to characterise, for certain classes of functionals, when scenario-basedness and $\mathfrak{p}$-invariance can coexist. Unsurprisingly, this restricts the possibilities for the choices of $\mathfrak{p}$ and $\mathfrak{Q}$ drastically.

For the following lemma, we assume a vector $\mathfrak{q} = (\mathfrak{q}_1, ..., \mathfrak{q}_r)$ of convex-ranged probability charges satisfies $\mathfrak{p}(N) = 0$ if and only if $\max_{1\leq s\leq r} \mathfrak{q}_s(N) = 0$, $N \in \Sigma$ (abbreviated by $\mathfrak{p} \approx \mathfrak{Q} = \{\mathfrak{q}_1, ..., \mathfrak{q}_r\}$). Moreover, by Lyapunov's Convexity Theorem, the range

$$\mathcal{R} := \{(\mathfrak{q}_1(A), ..., \mathfrak{q}_r(A), \mathfrak{p}(A)) \mid A \in \Sigma\} \subset [0,1]^{r+1}$$

is convex.

Lemma 7.2. *In the situation above, assume we can find $A, B \in \Sigma$ such that $\mathfrak{q}(A) = \mathfrak{q}(B)$, but $\mathfrak{p}(A) \neq \mathfrak{p}(B)$. Let $0 < p_1 < 1$ be arbitrary. Then there exists a nondecreasing sequence $(p_n)_{n\in\mathbb{N}} \subset]0,1]$ and a sequence $(\mathbf{q}_n)_{n\in\mathbb{N}} \subset [0,1]^r$ such that.*

(1) $\sup_{n\in\mathbb{N}} p_n = 1$.
(2) $\{(\mathbf{q}_n, p_n), (\mathbf{q}_n, p_{n+1}) \mid n \in \mathbb{N}\} \subset \mathcal{R}$.

Proof. The assumption implies that we find $0 < p < \widehat{p} < 1$ and $\mathbf{z} \in [0,1]^r$ such that

$$(\mathbf{z}, p), (\mathbf{z}, \widehat{p}) \in \mathcal{R}.$$

In particular, the convex hull of $\{(\mathbf{z}, p), (\mathbf{z}, \widehat{p}), (\mathbf{0}, 0), (\mathbf{1}, 1)\}$ is a subset of $\mathcal{R}$ and, for every $0 < t < 1$,

$$\exists\, s > t\, \exists\, u < t\, \exists\, \mathbf{q} \in [0,1]^r : \quad (\mathbf{q}, u), (\mathbf{q}, t), (\mathbf{q}, s) \in \mathcal{R}. \tag{7.1}$$

For the sake of brevity, we call a $\mathbf{q}$ as in (7.1) *admissible for t*. Fix $0 < p_1 < 1$ and set $\mathbf{q}_0 := \mathbf{0} \in [0,1]^r$. Having constructed $p_1, ..., p_k$ and $\mathbf{q}_0, ..., \mathbf{q}_{k-1}$, we iterate the construction in the following manner:

Case 1: $p_k = 1$. Set $p_{k+1} = p_k$ and $\mathbf{q}_k = \mathbf{q}_{k-1}$.

Case 2: $p_k < 1$. Set

$$\tau := \sup\{s - p_k \mid s > p_k,\ (s, \mathbf{q}) \in \mathcal{R} \text{ for some } \mathbf{q} \text{ admissible for } p_k\}.$$

Using (7.1), we find $p_{k+1} > p_k$ and $\mathbf{q}_k \in [0,1]^r$ admissible for p_k such that $(\mathbf{q}_k, p_k), (\mathbf{q}_k, p_{k+1}) \in \mathcal{R}$ and

$$\tfrac{1}{2}\tau < p_{k+1} - p_k \leq \tau.$$

Note that this construction leads to a nondecreasing sequence $(p_n)_{n\in\mathbb{N}} \subset]0,1]$ and a sequence $(\mathbf{q}_n)_{n\in\mathbb{N}} \subset [0,1]^r$. Define $p_\infty := \sup_{n\in\mathbb{N}} p_n$ and assume towards a contradiction that $p_\infty < 1$. Then we can find $\mathbf{q}_\infty$ admissible for p_∞. In particular, for n large enough, $\mathbf{q}_\infty$ is also admissible for p_n. By construction,

$$p_{n+1} - p_n \geq \tfrac{1}{2}(s - p_n)$$

for an arbitrary $s > p_\infty$ such that $(\mathbf{q}_\infty, s) \in \mathcal{R}$. As $n \to \infty$,

$$0 = \lim_{n\to\infty} p_{n+1} - p_n \geq \lim_{n\to\infty} \left(\tfrac{1}{2}(s - p_n)\right) = \tfrac{1}{2}(s - p_\infty) > 0.$$

This is impossible. The assumption that $p_\infty < 1$ must be absurd. $\square$

Lemma 7.2 seamlessly leads to the following auxiliary result.

Lemma 7.3. *Let $\mathfrak{p}, \mathfrak{q}_1, ..., \mathfrak{q}_r$ be convex-ranged probability charges on Σ such that $\mathfrak{p} \approx \mathfrak{Q} = \{\mathfrak{q}_1, ..., \mathfrak{q}_r\}$. Suppose $\mu\colon \Sigma \longrightarrow \mathbb{R}$ is a set function that is both $\mathfrak{p}$-invariant and $\mathfrak{Q}$-based.*[6] *Then one of the following alternatives holds:*

1. $\mathbf{q}(A) = \mathbf{q}(B) \implies \mathfrak{p}(A) = \mathfrak{p}(B)$.
2. *There exists $\alpha, \beta, \gamma \in \mathbb{R}$ such that*

$$\mu(A) = \begin{cases} \alpha & \mathfrak{p}(A) = 0 \\ \beta & 0 < \mathfrak{p}(A) < 1 \\ \gamma & \mathfrak{p}(A) = 1. \end{cases}$$

Proof. Suppose that alternative (1) fails. As μ is $\mathfrak{p}$-invariant, we only have to prove that $\mu(A) = \mu(B)$ for all $A, B \in \Sigma$ with $0 < \mathfrak{p}(A) < \mathfrak{p}(B) < 1$.
Set $p_1 := \mathfrak{p}(A)$ and let $(p_k)_{k\in\mathbb{N}} \subset]0, 1]$ and $(\mathbf{q}_k)_{k\in\mathbb{N}} \subset [0, 1]^r$ be the sequences of probabilities constructed in Lemma 7.2. A slight alteration of the construction allows us to select the sequence such that

$$N := \min\{k \in \mathbb{N} \mid p_k = \mathfrak{p}(B)\} < \infty.$$

Iteratively, for all $1 \le i \le N-1$ we find events $A_i, B_i \in \Sigma$ such that $\mathfrak{p}(A_i) = p_i$ and $\mathbf{q}(A_i) = \mathbf{q}_i$, while $\mathfrak{p}(B_i) = p_{i+1}$ and $\mathbf{q}(B_i) = \mathbf{q}_i$. Hence, using $\mathfrak{p}$-invariance and $\mathfrak{Q}$-basedness alternately,

$$\mu(A) = \mu(A_1) = \mu(B_1) = \mu(A_2) = ... = \mu(B_{N-1}) = \mu(B).$$

□

The set function μ in Lemma 7.3 is defined on events only. The result transfers to functionals $\varphi\colon \mathcal{X} \to \mathbb{R}$ though that are fully determined by their restriction to indicator functions $\mathbf{1}_A$, $A \in \Sigma$, such as Choquet integrals. A *capacity* is a set function $\mu\colon \Sigma \longrightarrow [0, 1]$ that is nondecreasing with respect to set inclusion and satisfies $\mu(\varnothing) = 1 - \mu(\Sigma) = 0$. The *Choquet integral* φ of $f \in \mathcal{X}$ with respect to μ is given by

$$\varphi(f) = \int_0^\infty \mu(f > x)dx + \int_{-\infty}^0 (1 - \mu(f > x))\, dx.$$

The integrals on the right-hand side are Riemann integrals. For more information on capacities and Choquet integrals we refer to the survey [20] and the monograph [9]. $\mathfrak{p}$-invariance of μ is equivalent to $\mathfrak{p}$-invariance of its associated Choquet integral viewed as a functional on $\mathcal{X}$ or $\mathcal{X}_0$. Similarly, $\mathfrak{Q}$-basedness of μ is equivalent to $\mathfrak{Q}$-basedness of its associated Choquet integral. Lemma 7.3 directly implies:

Corollary 7.4. *Let $\mathfrak{p}, \mathfrak{q}_1, ..., \mathfrak{q}_r$ be convex-ranged probability charges on Σ such that $\mathfrak{p} \approx \mathfrak{Q} = \{\mathfrak{q}_1, ..., \mathfrak{q}_r\}$ and let $\varphi\colon \mathcal{X} \longrightarrow \mathbb{R}$ be the Choquet integral with respect to a capacity μ. If φ is both $\mathfrak{p}$-invariant and $\mathfrak{Q}$-based, then one of the following alternatives holds:*

1. $\mathbf{q}(A) = \mathbf{q}(B) \implies \mathfrak{p}(A) = \mathfrak{p}(B)$.

[6] μ is here interpreted as a functional on $\mathcal{D} := \{\mathbf{1}_A \mid A \in \Sigma\}$.

(2) *There exists* $0 \leq \beta \leq 1$ *such that*

$$\mu(A) = \begin{cases} 0 & \mathfrak{p}(A) = 0 \\ \beta & 0 < \mathfrak{p}(A) < 1 \\ 1 & \mathfrak{p}(A) = 1. \end{cases}$$

Corollary 7.4 can be seen as a counterpart to Theorem 4.1 and Corollary 5.1 in its comparison of $\mathfrak{p}$-invariance and $\mathfrak{Q}$-basedness. Its drawback is that the functionals to which it applies are quite specific. Whether and how the statement can be extended remains open to us.

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