Uniform Rotundity and Best Approximation

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February 12, 2024

Abstract

Working constructively throughout, we prove that if K is an inhabited, complete, uniformly rotund subset of a normed space X, L is a located convex subset of X containing at least two distinct points, and $d \equiv \inf_{x \in K} \rho(x, L)$ exists, then there exists a strongly unique point $x_\infty \in K$ such that $\rho(x_\infty, L) = d$. To do so, we introduce the notion of sufficient convexity for real-valued functions on a metric space, and discuss the attainment of the infimum of such a function when that infimum exists.

Keywords: sufficiently convex functions, uniform rotundity, separation theorem for convex sets

The framework of this paper is Bishop-style constructive mathematics (**BISH**), which, for all practical purposes, can be viewed as mathematics developed using intuitionistic logic and based on an appropriate foundation such as CZF [1], Martin-Löf type theory [8, 9], or constructive Morse set theory [5]. Thus all our proofs embody algorithms that can be extracted for computer implementation (see, for example, [7, 10, 11]).

We call a mapping f of a metric space X into $\mathbf{R}$ *sufficiently convex* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, x' \in X$ with $\rho(x, x') > \varepsilon$, there exists $z \in X$ such that $f(z) + \delta < \max\{f(x), f(x')\}$. Here ρ denotes the metric on X.

Proposition 1 *The following are equivalent conditions on a mapping* f *of a metric space* X *into* $\mathbf{R}$*, such that* $\mu \equiv \inf f$ *exists.*

(i) f *is sufficiently convex.*

(ii) for each $\varepsilon > 0$ *there exists* $\tilde{\delta} > 0$ *such that if* $x, x' \in X$*,* $f(x) < \mu + \tilde{\delta}$*, and* $f(x') < \mu + \tilde{\delta}$*, then* $\rho(x, x') < \varepsilon$*.*

Proof. First suppose that f is sufficiently convex. Given $\varepsilon > 0$, pick $\delta > 0$ such that if $x, x' \in X$ and $\rho(x, x') > \varepsilon/2$, then $f(z) + \delta < \max\{f(x), f(x')\}$ for some $z \in X$. Let $\tilde{\delta} := \delta$ and consider $x, x' \in X$ such that $f(x) < \mu + \delta$, and $f(x') < \mu + \delta$. If $\rho(x, x') > \varepsilon/2$, then there exists $z \in X$ such that

$$f(z) + \delta < \max\{f(x), f(x')\} < \mu + \delta$$

and therefore $f(z) < \mu$, which is absurd. Hence $\rho(x, x') \leq \varepsilon/2 < \varepsilon$.

Conversely, suppose that f satisfies condition (ii). Given $\varepsilon > 0$, choose $\tilde{\delta}$ as in that condition. If $x, x' \in X$ and $\rho(x, x') > \varepsilon$, then $\max\{f(x), f(x')\} \geq \mu + \tilde{\delta}$. By the definition of μ, there exists $z \in X$ such that

$$f(z) < \mu + \frac{\tilde{\delta}}{2}$$

and hence

$$f(z) + \frac{\tilde{\delta}}{2} < \mu + \tilde{\delta} \leq \max\{f(x), f(x')\}.$$

Therefore, we may set $\delta := \frac{\tilde{\delta}}{2}$. □

The following result is was communicated to us by Peter Aczel many years ago.

Proposition 2 *Let* X *be a complete metric space, and let* f *be a sequentially continuous sufficiently convex mapping of* X *into* $\mathbf{R}$ *such that* $\mu \equiv \inf f$ *exists. Then there exists* $\xi \in X$ *such that* $f(\xi) = \mu$. *Moreover, if* $x \in X$ *and* $x \neq \xi$, *then* $f(x) > \mu$.

Proof. In view of Proposition 1, we can construct a strictly decreasing sequence $(\delta_n)_{n \geqslant 1}$ of positive numbers such that for each n, if $x, x' \in X$, $f(x) < \mu + \delta_n$, and $f(x') < \mu + \delta_n$, then $\rho(x, x') < 2^{-n}$. For each n, pick $x_n \in X$ such that $f(x_n) < \mu + \delta_n$. Then $\rho(x_m, x_n) < 2^{-n}$ for all $m \geqslant n$, so $(x_n)_{n \geqslant 1}$ is a Cauchy sequence in X. Since X is complete, $\xi \equiv \lim_{n \to \infty} x_n$ exists in X. By the sequential continuity of f, $\mu \leq f(\xi) \leq \mu$, so $f(\xi) = \mu$. Moreover, if $x \in X$ and $\rho(x, \xi) > 0$, then, with $\varepsilon := \frac{1}{2}\rho(x, \xi)$ and $\delta > 0$ as in the definition of 'sufficiently convex', there exists $z \in X$ such that

$$\mu < \mu + \delta \leq f(z) + \delta < \max\{f(\xi), f(x)\} = \max\{\mu, f(x)\} = f(x).$$

□

A subset L of a metric space is *located* if for all $x \in X$ the distance

$$\rho(x, L) := \inf\{\rho(x, y) \mid y \in L\}$$

exists.

Lemma 3 *Let* L *be an inhabited, located, and convex subset of a normed space* X. *Then for all* x, x' *in* X *and* $t \in [0, 1]$,

$$\rho(tx + (1-t)x', L) \leq t\rho(x, L) + (1-t)\rho(x', L).$$

Proof. Given $x, x' \in X$, $t \in [0,1]$, and $\varepsilon > 0$, pick $y, y' \in L$ such that

$$\|x - y\| < \rho(x, L) + \varepsilon \text{ and } \|x' - y'\| < \rho(x', L) + \varepsilon.$$

Then

$$\begin{aligned}
\rho(tx + (1-t)x', L) &\leq \|tx + (1-t)x' - ty - (1 - ty')\| \\
&\leq t\,\|x - y\| + (1-t)\,\|x' - y'\| \\
&\leq t\rho(x, L) + (1-t)\rho(x', L) + t\varepsilon + (1-t)\varepsilon \\
&\leq t\rho(x, L) + (1-t)\rho(x', L) + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the result follows. □

A normed space X is *uniformly convex* if for each $\varepsilon > 0$ there exists δ with $0 < \delta < 1$ such that if x, y are elements of X with $\|x\| = 1 = \|y\|$ and $\|x - y\| \geq \varepsilon$, then $\left\|\frac{1}{2}(x+y)\right\| \leq \delta$. Hilbert spaces, and L_p spaces with $p > 1$, are uniformly convex [4, page 322, Corollary (3.22)].

Lemma 4 *Let X be a uniformly convex normed space. Then for all $\tilde{\varepsilon} > 0$ and $M > 0$ there exists $\tilde{\delta} > 0$ such that if x, y are elements of X with $\|x\| = \|y\| \leq M$ and $\|x - y\| \geq \tilde{\varepsilon}$, then $\|\frac{1}{2}(x+y)\| + \tilde{\delta} \leq \|x\|$.*

Proof. Let $\tilde{\varepsilon} > 0$ and consider any $x, y \in X$ such that $\|x\| = \|y\| \leq M$ and $\|x - y\| \geq \tilde{\varepsilon}$. As $\tilde{\varepsilon} \leq \|x - y\| \leq 2\|x\|$, we infer $\|x\| = \|y\| \geq \tilde{\varepsilon}/2 > 0$. Set $\varepsilon := \frac{\tilde{\varepsilon}}{M}$ and compute $\delta \in (0,1)$ as in the definition of uniform convexity. As $x/\|x\|$ and $y/\|y\|$ are unit vectors with

$$\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| = \frac{1}{\|x\|}\|x - y\| \geq \frac{\tilde{\varepsilon}}{M} = \varepsilon,$$

we obtain

$$\frac{1}{\|x\|}\left\|\frac{1}{2}(x+y)\right\| \leq \delta.$$

Hence, using that $\|x\| \geq \tilde{\varepsilon}/2$,

$$\left\|\frac{1}{2}(x+y)\right\| \leq \delta\|x\| \leq \|x\| - (1-\delta)\|x\| \leq \|x\| - (1-\delta)\frac{\tilde{\varepsilon}}{2}.$$

Set $\tilde{\delta} := (1-\delta)\frac{\tilde{\varepsilon}}{2}$. □

Lemma 5 *Let X be a uniformly convex normed space, and let $K \subset X$ be inhabited, convex, and norm bounded. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, x' \in K$ with $\|x - x'\| \geq \varepsilon$ we have $\left\|\frac{1}{2}(x+x')\right\| + \delta \leq \max\{\|x\|, \|x'\|\}$. In particular $f(x) = \|x\|$, $x \in K$, defines a sufficiently convex function.*

Proof. Let $\varepsilon > 0$ and let $M > 0$ be a norm bound for K. For $\tilde{\varepsilon} := \varepsilon/2$ and M compute $\tilde{\delta} > 0$ as in Lemma 4. Choose $\delta > 0$ with $\delta < \min\{\varepsilon/4, \tilde{\delta}/2\}$ and consider $x, x' \in K$ with $\|x - x'\| \geq \varepsilon$. Either $\big|\|x\| - \|x'\|\big| > \delta$ or $\big|\|x\| - \|x'\|\big| < 2\delta$. In the first case note that $\min\{\|x\|, \|x'\|\} < \max\{\|x\|, \|x'\|\} - \delta$ and thus

$$\left\|\frac{1}{2}(x + x')\right\| \leq \frac{1}{2}(\max\{\|x\|, \|x'\|\} + \min\{\|x\|, \|x'\|\}) < \max\{\|x\|, \|x'\|\} - \frac{\delta}{2}.$$

Now assume the second case. Then by the triangle inequality,

$$\varepsilon \leq \|x - x'\| \leq 2(\|x\| + \delta) \quad \text{and} \quad \varepsilon \leq \|x - x'\| \leq 2(\|x'\| + \delta)$$

implying that $\min\{\|x\|, \|x'\|\} > 0$. Consider $y := \frac{\|x\|}{\|x'\|}x'$, and note that

$$\|x' - y\| = \big|\|x'\| - \|x\|\big| < 2\delta, \quad \|y\| = \|x\| \leq M,$$

and

$$\|x - y\| \geq \|x - x'\| - \|x' - y\| > \varepsilon - 2\delta > \frac{\varepsilon}{2} = \tilde{\varepsilon}.$$

By choice of $\tilde{\delta}$ we have

$$\begin{aligned} \|x\| &\geq \frac{1}{2}\|x + y\| + \tilde{\delta} \geq \frac{1}{2}(\|x + x'\| - \|x' - y\|) + \tilde{\delta} \\ &> \frac{1}{2}\|x + x'\| - \delta + \tilde{\delta} > \frac{1}{2}\|x + x'\| + \delta. \end{aligned}$$

As $\|x\| \leq \max\{\|x\|, \|x'\|\}$, the lemma is proved. □

Theorem 6 *Let X be a uniformly convex normed space, and let $K \subset X$ be an inhabited, complete, and convex set. Moreover, let $y \in X$ and assume that*

$$\mu := \inf\{\|y - x\| : x \in K\}$$

exists. Then there exists $x_0 \in K$ such that $\|y - x_0\| = \mu$. If $x' \in K$ such that $x' \neq x_0$, then $\|y - x'\| > \mu$.

Proof. As the algebraic difference $K - \{y\}$ inherits all properties from K, we may assume that $y = 0$. Pick $z \in K$. Then

$$\mu = \inf\{\|x\| : x \in K, \|x\| \leq M\}$$

where $M > 0$ satisfies $M > \|z\|$. The set $\tilde{K} := \{x \in K : \|x\| \leq M\}$ is inhabited, convex, bounded, and complete. Therefore, the mapping $x \mapsto \|x\|$ on $\tilde{K}$ is sufficiently convex by Lemma 5 and has a unique minimum point $x_0 \in \tilde{K}$ by Proposition 2. □

An immediate consequence of Theorem 6 is the proof of [4, Problem 11, p. 391], namely:

Corollary 7 *Let* B *be a uniformly convex Banach space, and let* $K \subset B$ *be a closed, located, and convex set. Then each* $y \in B$ *has a unique closest point* $x_0 \in K$*, i.e.* $\|y - x_0\| = \rho(y, K)$*, and if* $x' \in K$ *is such that* $x' \neq x_0$*, then* $\|y - x'\| > \rho(y, K)$.

A subset C of a normed space X is *uniformly rotund* if it is convex and for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, x' \in C$ and $\|x - x'\| \geq \varepsilon$, then $\frac{1}{2}(x + x') + z \in C$ for all $z \in X$ with $\|z\| \leq \delta$.

Proposition 8 *A normed linear space* X *is uniformly convex if and only if its closed unit ball* B *is uniformly rotund.*

Proof. Suppose that X is uniformly convex, and let $\varepsilon > 0$. Compute $\delta > 0$ for ε and $K = B$ as in Lemma 5. Then for all $x, x' \in B$ such that $\|x - x'\| \geq \varepsilon$ and any $z \in X$ with $\|z\| \leq \delta$ it follows that

$$\left\|\frac{1}{2}(x + x') + z\right\| \leq \left\|\frac{1}{2}(x + x')\right\| + \delta \leq \max\{\|x\|, \|x'\|\} \leq 1.$$

Hence, $\frac{1}{2}(x + x') + z \in B$, so B is uniformly rotund.

Conversely, suppose that B is uniformly rotund, let $\varepsilon > 0$, and choose $\delta < 1$ as in the definition of uniformly rotund. If x, y are unit vectors of X with $\|x - y\| \geq \varepsilon$, then $\left\|\frac{1}{2}\delta(x + y)\right\| \leq \delta$, so

$$(1 + \delta)\left\|\tfrac{1}{2}(x + y)\right\| = \left\|\tfrac{1}{2}(x + y) + \tfrac{1}{2}\delta(x + y)\right\| \leq 1$$

and therefore $\left\|\frac{1}{2}(x + y)\right\| \leq (1 + \delta)^{-1} < 1$. □

Proposition 9 *Let* K *be an inhabited and uniformly rotund subset of a normed space* X*, and* L *an inhabited, located, and convex subset of* X *that is disjoint from* K*. Then* $f(x) \equiv \rho(x, L)$ *defines a sufficiently convex function on* K.

Proof. For $\varepsilon > 0$ let $\delta > 0$ as in the definition of 'uniform rotundity' for K, and let $\xi := \delta/2$. Consider $x, x' \in K$ such that $\|x - x'\| \geq \varepsilon$. Let $u := \frac{1}{2}(x + x')$ and fix $v \in L$ such that $\|v - u\| < \rho(u, L) + \xi$. Note that $\|v - u\| \geq \delta$, because by choice of δ, if we had $\|v - u\| < \delta$, then $v = u + (v - u) \in K$ which is absurd since K and L are disjoint. Let

$$z := u + \frac{\delta}{\|v - u\|}(v - u).$$

Then $\|z - u\| = \delta$, and therefore $z = u + (z - u) \in K$. Recalling Lemma 3, we have

$$\begin{aligned} f(z) + \xi &\leq \|v - z\| + \xi = \left(1 - \frac{\delta}{\|u - v\|}\right)\|v - u\| + \xi \\ &= \|v - u\| - \xi < f(u) \leq \max\{f(x), f(x')\}. \end{aligned}$$

□

To see that in Proposition 9 we cannot replace uniform rotundity by mere convexity, take X to be the Euclidean plane $\mathbf{R}^2$, $K = \{(a, b) \in \mathbf{R}^2 : a \leq 0\}$, and $L = \{(a, b) \in \mathbf{R}^2 : a \geq 1\}$; we have

$$\inf_{x \in K} \rho(x, L) = 1 = \|(0, b) - (1, b)\|$$

for all $b \in \mathbf{R}$, so, in view of Proposition 2, $x \mapsto \rho(x, L)$ is not sufficiently convex on K.

Recall here *Bishop's Lemma* [6, Proposition 3.1.1]:

> Let Y be an inhabited, complete, located subset of a metric space X. Then for each $x \in X$ there exists $y \in Y$ such that if $x \neq y$, then $\rho(x, Y) > 0$.

Theorem 10 *Let K be an inhabited, complete, and uniformly rotund subset of a normed space X, and L an inhabited, located, and convex subset of X that is disjoint from K. Suppose also that $d \equiv \inf_{x \in K} \rho(x, L)$ exists. Then there exists $\xi \in K$ such that* (i) $\rho(\xi, L) = d$ *and (ii) $\rho(x, L) > d$ for all $x \in K$ with $x \neq \xi$. If, in addition, L is complete, then there exists $y \in L$ such that if $\xi \neq y$, then $d > 0$.*

Proof. By Proposition 9, $f(x) \equiv \rho(x, L)$ defines a sufficiently convex, function on K. Since K is complete and d exists, Proposition 2 produces $\xi \in K$ with properties (i) and (ii). If also L is complete, then we complete the proof by invoking Bishop's Lemma. □

Lemma 11 *Let Y be an inhabited and convex subset of a Hilbert space H, and a a point of H such that $d = \rho(a, Y)$ exists. Then there exists $b \in \overline{Y}$ such that $\|a - b\| = d$. Moreover,*

(i) $\|a - y\| > d$ *whenever* $y \in \overline{Y}$ *and* $y \neq b$*;*

(ii) $\langle a - b, b - y \rangle \geq 0$*, and therefore* $\langle a - b, a - y \rangle \geq d^2$*, for all* $y \in Y$*.*

Proof. This is a well-known result on Hilbert space. For instance Lemma 1 in [2] proves the existence of $b \in \overline{Y}$ such that $\|a - b\| = d$ and (ii) holds. Conclusion (i) follows from (ii) since for all $y \in Y$

$$\|a-y\|^2 = \|a-b+b-y\|^2 = \|a-b\|^2 + \|b-y\|^2 + 2\langle a - b, b - y \rangle \geq d^2 + \|b-y\|^2.$$

□

Theorem 12 *Let K be an inhabited, closed, and uniformly rotund subset of a Hilbert space H, and L an inhabited, closed, located, and convex subset of H that is disjoint from K. Suppose also that $d \equiv \inf_{x \in K} \rho(x, L)$ exists. Then there exist $x_\infty \in K$ and $y_\infty \in L$ such that $\|x_\infty - y_\infty\| = d$. Moreover,*

(i) $\|x-y\| > d$ *whenever* $x \in K$ *and* $y \in L$ *and either* $x \neq x_\infty$ *or* $y \neq y_\infty$*;*

(ii) $\langle x_\infty - y_\infty, y_\infty - y\rangle \geq 0$*, and therefore* $\langle x_\infty - y_\infty, x_\infty - y\rangle \geq d^2$*, for all* $y \in L$*.*

Proof. By Theorem 10, there exists $x_\infty \in K$ such that $d = \rho(x_\infty, L)$. By Lemma 11 there exists $y_\infty \in L$ such that $\|x_\infty - y_\infty\| = \rho(x_\infty, L)$ and properties (i) and (ii) hold. □

Note that also in Theorem 12 we cannot replace uniformly rotundity by mere convexity: Consider $H = \mathbf{R}^2$ and $K = \{(a,b) \in \mathbf{R}^2 : b \geq e^a + 1\}$ and $L = \{(a,b) \in \mathbf{R}^2 : b \leq -e^a - 1\}$. Then $d = 2$, but there is no $x \in K$ and $y \in L$ such that $\|x-y\| = 2$.

Theorem 12 leads us to a new constructive separation theorem where the separating linear functional is constructed as the difference of the points of closest distance.

Theorem 13 *Let* K *be an inhabited, closed, located, and uniformly rotund subset of a Hilbert space* H*, and* L *an inhabited, closed, located, and convex subset of* H*. Suppose that* $d \equiv \inf_{x\in K} \rho(x, L)$ *exists and is positive, let* $x_\infty \in K$ *and* $y_\infty \in L$ *be as in Theorem 12, and let* $p = x_\infty - y_\infty$*. Then*

$$\langle p, x-y\rangle \geq d^2 \quad \textit{for all } x \in K \textit{ and } y \in L.$$

The normed linear functional $u(x) = \langle d^{-1}p, x\rangle$*,* $x \in H$*, satisfies* $\|u\| = 1$ *and* $u(x) \geq u(y) + d$ *for all* $x \in K$ *and* $y \in L$*. In particular* $u(x_\infty) \leq u(x)$ *for all* $x \in K$*, where* $u(x_\infty) < u(x)$ *if* $x \neq x_\infty$*, and* $u(y_\infty) \geq u(y)$ *for all* $y \in L$*.*

Proof. Construct $x_\infty \in K$ and $y_\infty \in L$ as in Theorem 12, and let

$$p = x_\infty - y_\infty.$$

Then, by Theorem 12, for all $y \in Y$ we have

$$\langle p, x_\infty - y\rangle = \langle x_\infty - y_\infty, x_\infty - y\rangle \geq d^2.$$

On the other hand, since K is located Lemma 11 provides the existence of a unique $b \in K$ such that $\rho(y_\infty, K) = \|y_\infty - b\|$. As $\rho(y_\infty, K) = d = \|y_\infty - x_\infty\|$ it follows that indeed $b = x_\infty$ and thus by Lemma 11 that

$$\langle y_\infty - x_\infty, x_\infty - x\rangle \geq 0 \tag{1}$$

for all $x \in K$. Hence, for $x \in K$ and $y \in L$,

$$\begin{aligned}\langle p, x-y\rangle &= \langle p, x_\infty - y\rangle + \langle p, x - x_\infty\rangle \\ &\geq d^2 + \langle x_\infty - y_\infty, x - x_\infty\rangle \\ &= d^2 + \langle y_\infty - x_\infty, x_\infty - x\rangle \geq d^2.\end{aligned}$$

As regards the properties of u, note that $u(x_\infty) \leq u(x)$ for all $x \in K$ follows from (1) and $u(y_\infty) \geq u(y)$ for all $y \in L$ is shown in Theorem 12 (ii). Let $x \in K$ such that $x \neq x_\infty$. Then by uniform rotundity of K there is $\delta > 0$ such that $\frac{1}{2}(x_\infty + x) + z \in K$ for all $z \in H$ with $\|z\| \leq \delta$. Let $z := -\frac{\delta}{d}p$. Then $\|z\| = \delta$ and therefore $\frac{1}{2}(x_\infty + x) + z \in K$. It follows that $u(\frac{1}{2}(x_\infty + x) + z) \geq u(x_\infty)$, and thus $u(x) + 2u(z) \geq u(x_\infty)$. As $u(z) = -\frac{\delta}{d^2}\langle p, p\rangle = -\delta < 0$, we conclude that $u(x) > u(x_\infty)$. □

By Theorem 13 we may construct supporting hyperplanes $P_K := \{x \in H : u(x) = u(x_\infty)\}$ of K and $P_L := \{x \in H : u(x) = u(y_\infty)\}$ of L, respectively, where P_K intersects with K in the unique point x_∞, and P_L instersects with L in y_∞. The uniqueness of the intersection point x_∞ of P_K and K is strong, in the sense that any point $x \in K$ distinct from x_∞ is bounded away from P_K since $u(x) > u(x_\infty)$.

In trying to apply the foregoing theorems, it is natural to think of the case where the uniformly rotund set K is compact. In that case, if K is nontrivial, Corollary 15 below shows that H is finite-dimensional.

Proposition 14 *Let X be a normed space, and S be a uniformly rotund subset of X that contains two distinct points. Then S contains an open ball of positive radius.*

Proof. Let a, b be two distinct points of S. There exists $\delta > 0$ such that if $x, y \in S$ and $\|x - y\| \geq \|a - b\|$, then $\frac{1}{2}(x + y) + z \in S$ for all $z \in X$ with $\|z\| \leq \delta$. Consider the open ball $B(\frac{1}{2}(a + b), \delta)$ of radius δ with center $\frac{1}{2}(a + b)$. If $z \in B(\frac{1}{2}(a + b), \delta)$, then $\|z - \frac{1}{2}(a + b)\| < \delta$ and thus

$$z = \tfrac{1}{2}(a + b) + \left(z - \tfrac{1}{2}(a + b)\right) \in S,$$

so $B(\frac{1}{2}(a + b), \delta)$ is the required ball. □

Corollary 15 *A normed space that has a totally bounded and uniformly rotund subset which contains two distinct points is finite-dimensional.*

Proof. This follows from the preceding proposition and [6, Proposition 4.1.13]. □

References

[1] P. Aczel and M. Rathjen, *Notes on Constructive Set Theory*, `http://www1.maths.leeds.ac.uk/~rathjen/book.pdf`.

[2] J. Berger and G. Svindland, *A separating hyperplane theorem, the fundamental theorem of asset pricing, and Markov's principle,* Ann. Pure and Applied Logic **167**, 1161–1171, 2016.

[3] E. Bishop, *Foundations of Constructive Analysis*, McGraw-Hill, New York, 1967.

[4] E. Bishop and D.S. Bridges, *Constructive Analysis*, Grundlehren der math. Wiss. **279**, Springer-Verlag, Heidelberg, 1985.

[5] D.S. Bridges, *Morse set theory as a foundation for constructive mathematics,* Theoretical Comp. Sci. **928**, 115–135, 2022.

[6] D.S. Bridges and L.S. Vîţă, *Techniques of Constructive Analysis,* Springer New York, 2006.

[7] R.L. Constable et al., *Implementing Mathematics with the Nuprl Proof Development System*, Prentice-Hall, Englewood Cliffs, New Jersey, 1986.

[8] P. Martin-Löf, *Intuitionistic type theory* (Notes by Giovanni Sambin of a series of lectures given in Padua, June 1980), Bibliopolis, Napoli, 1984.

[9] G. Sambin and J. Smith (eds), *Twenty-five Years of Constructive Type Theory*, Oxford Logic Guides **36**, The Clarendon Press, Oxford, 1998.

[10] H. Schwichtenberg, Computational Aspects of Bishop's Constructive Mathematics, in: D. Bridges, H. Ishihara, M. Rathjen, and H. Schwichtenberg (Eds.), *Handbook of Constructive Mathematics* (Encyclopedia of Mathematics and its Applications), 715–748, Cambridge University Press., 2023. doi:10.1017/9781009039888.027

[11] H. Schwichtenberg and S.S. Wainer, *Proofs and Computations.* Perspectives in Logic. Association for Symbolic Logic and Cambridge University Press, 2012