From Ruin Theory to Solvency in Non-Life Insurance

Mario V. Wüthrich

RiskLab ETH Zurich &

Swiss Finance Institute SFI

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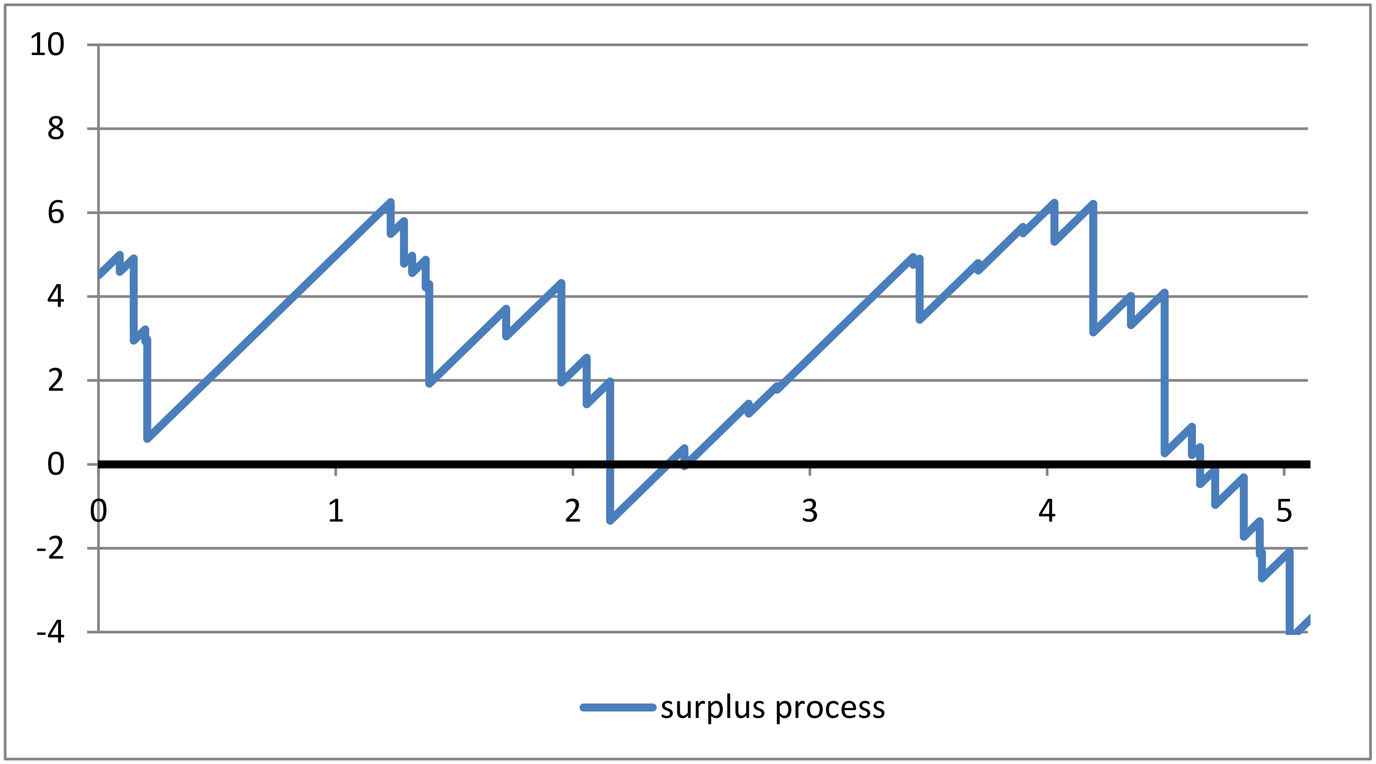
Aim of this presentation

We start from Lundberg's thesis (1903) on ruin theory and modify his model step by step until we arrive at today's solvency considerations.



Cramér-Lundberg model

Consider the surplus process $(C_t)_{t\geq 0}$ given by

$$C_t = c_0 + \pi t - \sum_{i=1}^{N_t} Y_i,$$

where

$$c_0 \geq 0 \qquad \text{initial capital,}$$

$$\pi > 0 \qquad \text{premium rate,}$$

$$L_t = \sum_{i=1}^{N_t} Y_i \geq 0 \qquad \text{homogeneous compound Poisson claims process,}$$

Harald Cramér

satisfying the net profit condition (NPC): $\pi > \mathbb{E}[L_1]$.

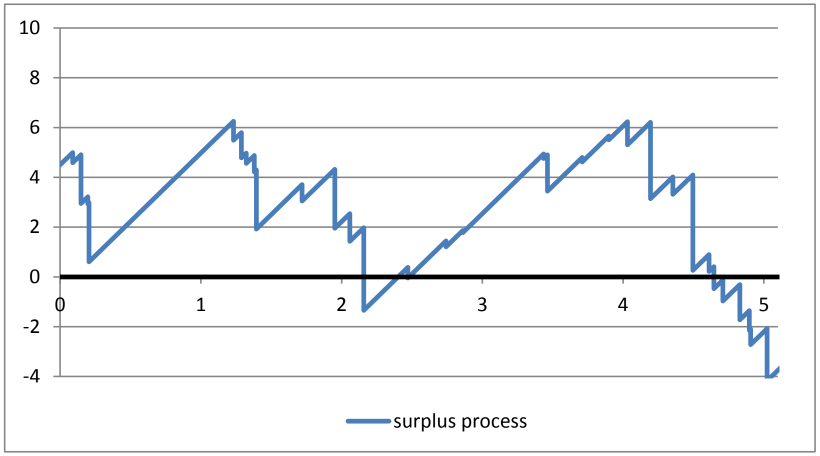
Ultimate ruin probability

The ultimate ruin probability for initial capital $c_0 \geq 0$ is given by

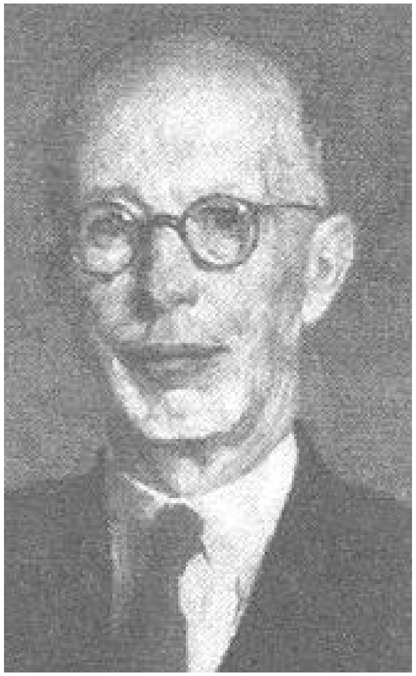
$$\psi(c_0) = \mathbb{P}\left[\inf_{t \in \mathbb{R}_+} C_t < 0 \middle| C_0 = c_0\right] = \mathbb{P}_{c_0}\left[\inf_{t \in \mathbb{R}_+} C_t < 0\right],$$

i.e. this is the infinite time horizon ruin probability.

Under (NPC):
$\psi(c_0) < 1$ for all $c_0 \geq 0$.



Lundberg's exponential bound



Filip Lundberg

Assume (NPC) and that the Lundberg coefficient $\gamma > 0$ exists. Then, we have exponential bound

$$\psi(c_0) \;\leq\; \exp\{-\gamma c_0\},$$

for all $c_0 \geq 0$ (large deviation principle (LDP)).

This is the light-tailed case, i.e. for the existence of $\gamma > 0$ we need *exponentially decaying* survival probabilities of the claim sizes Y_i,

because we require $\mathbb{E}[\exp\{\gamma Y_i\}] < \infty$.

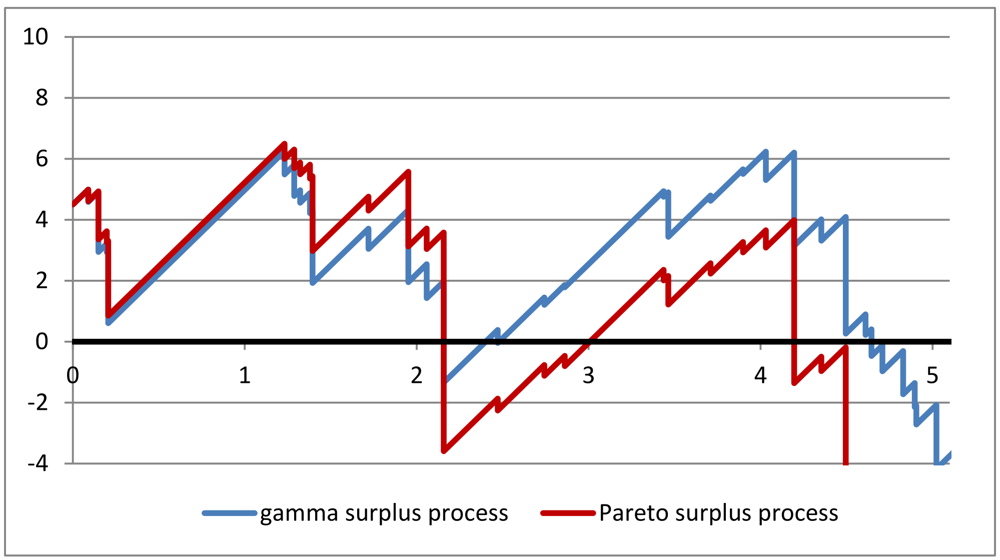
Subexponential case

Von Bahr, Veraverbeke, Embrechts investigate the heavy-tailed case.

In particular, for $Y_i \overset{\text{i.i.d.}}{\sim} \mathrm{Pareto}(\alpha > 1)$ and (NPC):

$$\psi(c_0) \sim \text{const } c_0^{-\alpha+1} \qquad \text{as } c_0 \to \infty.$$

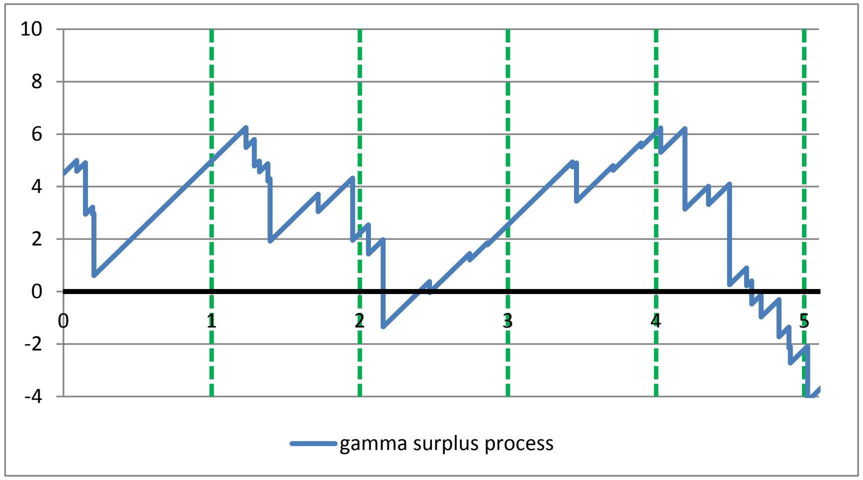
Heavy-tailed case provides a much slower decay.



Paul Embrechts

Discrete time ruin considerations

Insurance companies cannot continuously control their surplus processes $(C_t)_{t\geq 0}$.



They close their books and check their surplus on a *yearly time grid.*

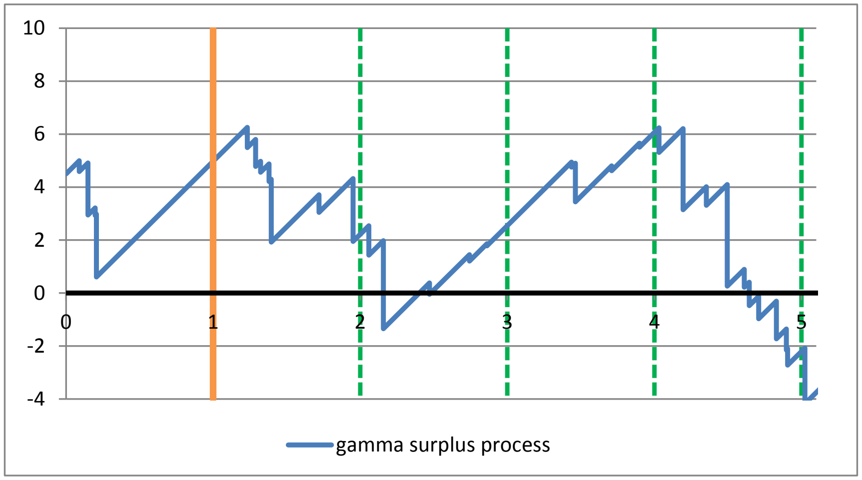
▷ Consider the discrete time ruin probability

$$\mathbb{P}_{c_0}\left[\inf_{n\in\mathbb{N}_0} C_n < 0\right] \;\leq\; \mathbb{P}_{c_0}\left[\inf_{t\in\mathbb{R}_+} C_t < 0\right] = \psi(c_0).$$

This leads to the study of the random walk $(C_n - c_0)_{n\in\mathbb{N}_0}$ for (discrete time) accounting years $n \in \mathbb{N}_0$.

One-period ruin problem

Insured buy *one-year* non-life insurance contracts: why bother about *ultimate* ruin probabilities?



Moreover, initial capital $c_0 \geq 0$ needs to be re-adjusted every accounting year.

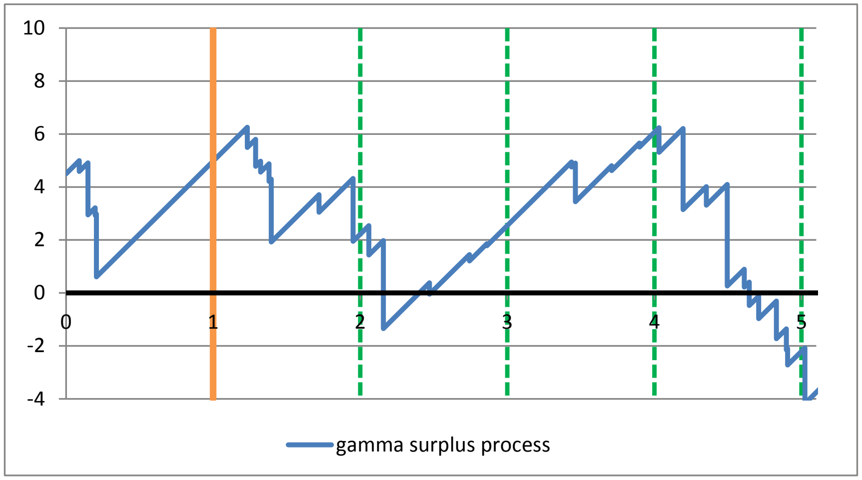
▷ Consider the (discrete time) one-year ruin probability

$$\mathbb{P}_{c_0}[C_1 < 0] \leq \mathbb{P}_{c_0}\left[\inf_{n \in \mathbb{N}_0} C_n < 0\right] \leq \mathbb{P}_{c_0}\left[\inf_{t \in \mathbb{R}_+} C_t < 0\right] = \psi(c_0).$$

This leads to the study of the surplus $C_1 = c_0 + \pi - \sum_{i=1}^{N_1} Y_i$ at time 1.

One-period problem and real world considerations

Why do we study so complex models when the real world problem is so simple?



- **Total asset value** at time 1: $A_1 = c_0 + \pi$.
- **Total liabilities** at time 1: $L_1 = \sum_{i=1}^{N_1} Y_i$.

$$C_1 = c_0 + \pi - \sum_{i=1}^{N_1} Y_i \; = \; A_1 - L_1 \; \overset{???}{\geq} \; 0. \qquad (1)$$

There are many modeling issues hidden in (1)! We discuss them step by step.

Value-at-Risk (VaR) risk measure

$$C_1 \;=\; A_1 - L_1 \;\overset{???}{\geq}\; 0.$$

Freddy Delbaen

▷ Value-at-Risk on confidence level $p = 99.5\%$ (Solvency II): choose c_0 minimal such that

$$\mathbb{P}_{c_0}[C_1 \geq 0] \;=\; \mathbb{P}[A_1 \geq L_1] \;=\; \mathbb{P}[L_1 - c_0 - \pi \leq 0] \;\geq\; p.$$

▷ Choose other (normalized) risk measures $\varrho : \mathcal{M} \subset L^1(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ and study

$$\varrho(L_1 - A_1) \;=\; \varrho\left(L_1 - c_0 - \pi\right) \;\overset{???}{\leq}\; 0,$$

where "$\leq$" implies **SOLVENCY** w.r.t. risk measure ϱ.

Asset return and financial risk (1/2)

- Initial capital at time 0: $c_0 \geq 0$.

- Premium received at time 0 for accounting year 1: $\pi > 0$.

▷ Total asset value at time 0: $a_0 = c_0 + \pi > 0$.

This asset value a_0 is *invested in different assets* $k \in \{1, \ldots, K\}$ at time 0.

asset classes

- cash and cash equivalents
- debt securities (bonds, loans, mortgages)
- real estate & property
- equity, private equity
- derivatives & hedge funds
- insurance & reinsurance assets
- other assets

Asset return and financial risk (2/2)

Choose an asset portfolio $\boldsymbol{x} = (x_1, \ldots, x_K)' \in \mathbb{R}^K$ at time 0 with initial value

$$a_0 = \sum_{k=1}^{K} x_k S_0^{(k)},$$

where $S_t^{(k)}$ is the price of asset k at time t. This provides value at time 1

$$A_1 = \sum_{k=1}^{K} x_k S_1^{(k)} = a_0 \left(1 + \boldsymbol{w}'\boldsymbol{R}_1\right),$$

for *buy & hold asset strategy* $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{x}) \in \mathbb{R}^K$ and (random) return vector $\boldsymbol{R}_1$.

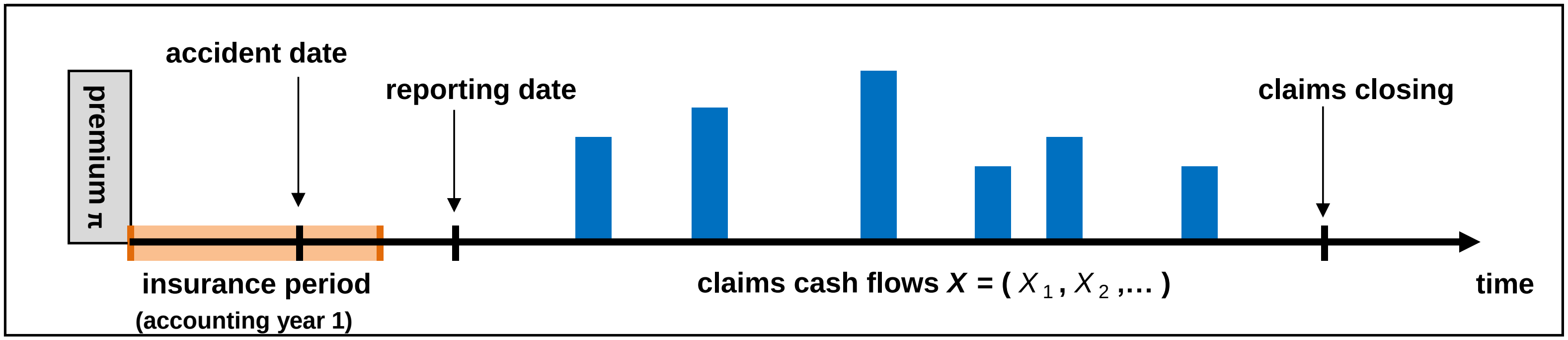
$$\varrho(L_1 - A_1) \;=\; \varrho\left(L_1 - a_0\left(1 + \boldsymbol{w}'\boldsymbol{R}_1\right)\right) \;\overset{???}{\leq}\; 0.$$

where "$\leq$" implies solvency w.r.t. risk measure ϱ and business plan $(L_1, a_0, \boldsymbol{w})$.

Insurance claim (liability) modeling (1/2)

MAIN ISSUE: modeling of insurance claim $L_1 = \sum_{i=1}^{N_1} Y_i$.

▷ Insurance claims are neither known nor can immediately be settled at occurrence!

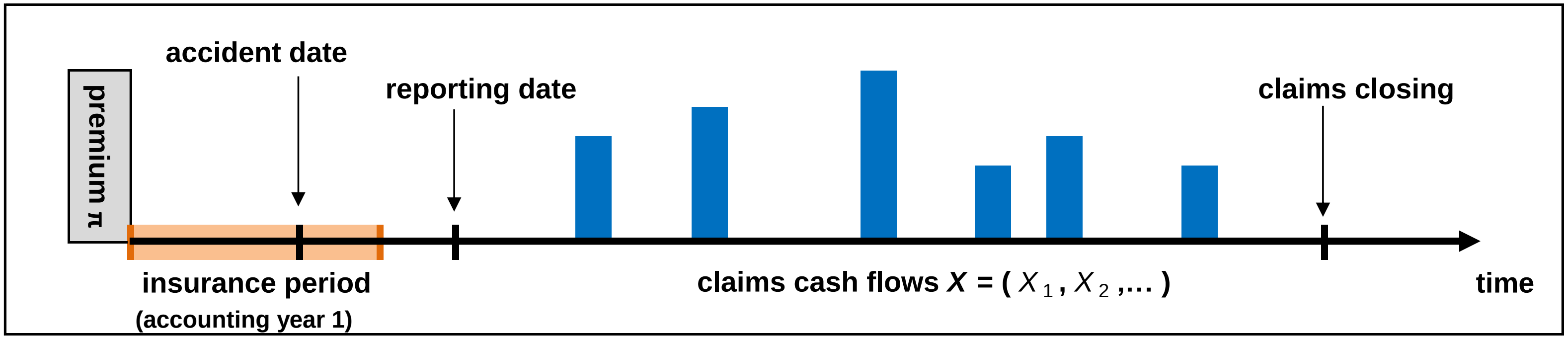


▷ Insurance claims of accounting year 1 generate insurance liability cash flow $\boldsymbol{X}$:

$$\boldsymbol{X} = (X_1, X_2, \ldots) \qquad \text{with } X_t \text{ being the payment in accounting year } t.$$

Question: How is the cash flow $\boldsymbol{X}$ related to the insurance claim L_1?

Insurance claim (liability) modeling (2/2)



▷ **Main tasks:**

- cash flow $\boldsymbol{X} = (X_1, X_2, \ldots)$ modeling,
- cash flow $\boldsymbol{X} = (X_1, X_2, \ldots)$ prediction,
- cash flow $\boldsymbol{X} = (X_1, X_2, \ldots)$ valuation,

using *all* available relevant information:

▷ exactly here the one-period problem turns into a multi-period problem.

Best-estimate reserves

Choose a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{N}_0}$ and

assume cash flow $\boldsymbol{X}$ is $\mathbb{F}$-adapted.

1st attempt to define L_1 (interpretation of Solvency II):

$$L_1 = X_1 + \sum_{s \geq 2} P(1, s) \, \mathbb{E}\left[X_s \middle| \mathcal{F}_1 \right],$$

where

- $\mathbb{E}\left[X_s \middle| \mathcal{F}_1 \right]$ is the best-estimate reserve (prediction) of X_s at time 1;
- $P(1, s)$ is the zero-coupon bond price at time 1 for maturity date s.

Note that L_1 is $\mathcal{F}_1$-measurable, i.e. observable w.r.t. $\mathcal{F}_1$ (information at time 1).

1st attempt to define L_1

$$L_1 = X_1 + \sum_{s \geq 2} P(1,s)\, \mathbb{E}\left[X_s \middle| \mathcal{F}_1 \right]. \tag{2}$$

Issue: Solvency II asks for economic balance sheet, but L_1 is *not* an economic value.

(a) Risk margin is missing:
any risk-averse risk bearer asks for such a (profit) margin.

(b) Zero-coupon bond prices and claims cash flows X_s, $s \geq 2$, may be influenced by the same risk factors and, thus, *there is no decoupling* such as (2).

2nd attempt to define L_1

Choose an appropriate **state-price deflator** $\boldsymbol{\varphi} = (\varphi_t)_{t \geq 1}$ and

$$L_1 = X_1 + \sum_{s \geq 2} \frac{1}{\varphi_1} \, \mathbb{E}\left[\varphi_s \; X_s \middle| \mathcal{F}_1 \right].$$

Hans Bühlmann

- $\boldsymbol{\varphi} = (\varphi_t)_{t \geq 1}$ is a strictly positive, a.s., and $\mathbb{F}$-adapted.
- $\boldsymbol{\varphi} = (\varphi_t)_{t \geq 1}$ reflects price formation at financial markets, in particular,

$$P(1, s) = \frac{1}{\varphi_1} \, \mathbb{E}\left[\varphi_s \middle| \mathcal{F}_1 \right].$$

- If φ_s and X_s are **positively correlated**, given $\mathcal{F}_1$, then

$$L_1 \;\; \geq \;\; X_1 + \sum_{s \geq 2} P(1, s) \, \mathbb{E}\left[X_s \middle| \mathcal{F}_1 \right].$$

Solvency at time 0

▷ Choose a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ such that it caries the random vectors $\boldsymbol{\varphi}$ (state-price deflator), $\boldsymbol{R}_1$ (returns of assets) and $\boldsymbol{X}$ (insurance liability cash flows) in a reasonable way.

▷ The business plan $(\boldsymbol{X}, a_0, \boldsymbol{w})$ is solvent w.r.t. the risk measure ϱ and state-price deflator $\boldsymbol{\varphi}$ if

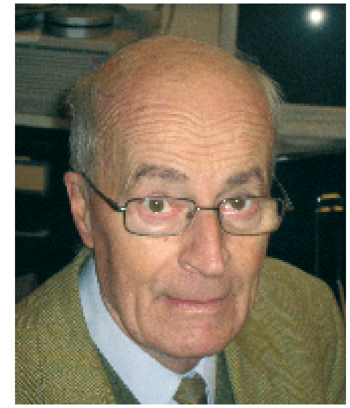
$$\varrho(L_1 - A_1) \;=\; \varrho\left(X_1 + \sum_{s \geq 2} \frac{1}{\varphi_1} \, \mathbb{E}\left[\varphi_s X_s \middle| \mathcal{F}_1\right] - a_0 \left(1 + \boldsymbol{w}' \boldsymbol{R}_1\right) \right) \;\leq\; 0.$$

Thus, it is likely (measured by ϱ and $\boldsymbol{\varphi}$) that the liabilities L_1 are covered by assets A_1 at time 1 in an economic balance sheet.

Acceptability arbitrage

- The choice of the state-price deflator $\boldsymbol{\varphi}$ and the risk measure ϱ cannot be done independently of each other:

  ⋆ $\boldsymbol{\varphi}$ describes the risk reward;
  ⋆ ϱ describes the risk punishment.



P. Artzner

- Assume there exist acceptable zero-cost portfolios $\mathbf{Y}$ with

$$\mathbb{E}[\boldsymbol{\varphi}'\mathbf{Y}] = 0 \quad \text{and} \quad \varrho\left(Y_1 + \sum_{s\geq 2} \frac{1}{\varphi_1}\, \mathbb{E}\left[\varphi_s Y_s \middle| \mathcal{F}_1\right]\right) < 0.$$

  Then, unacceptable positions can be turned into acceptable ones just by loading on more risk $\Longrightarrow$ acceptability arbitrage.

- Reasonable solvency models $(\boldsymbol{\varphi}, \varrho)$ should exclude acceptability arbitrage, see Artzner, Delbaen, Eisele, Koch-Medina.

Asset & liability management (ALM)

The business plan $(\mathbf{X}, a_0, \boldsymbol{w})$ is solvent w.r.t. risk measure ϱ and state-price deflator φ if

$$\varrho(L_1 - A_1) \;=\; \varrho\left(X_1 + \sum_{s \geq 2} \frac{1}{\varphi_1}\, \mathbb{E}\left[\varphi_s X_s \middle|\, \mathcal{F}_1\right] - a_0\left(1 + \boldsymbol{w}'\boldsymbol{R}_1\right)\right) \;\leq\; 0.$$

ALM optimize this business plan $(\mathbf{X}, a_0, \boldsymbol{w})$:
Which asset strategy $\boldsymbol{w} \in \mathbb{R}^K$ minimizes the capital $a_0 = c_0 + \pi$ and we still remain solvent?

▷ This is a non-trivial optimization problem.

▷ Of course, we need to exclude acceptability arbitrage, which may also provide restrictions on the possible asset strategies $\boldsymbol{w} \Longrightarrow$ eligible assets.

Summary of modeling tasks

- Provide reasonable stochastic models for $\boldsymbol{R}_1$, $\boldsymbol{X}$ and φ (yield curve extrapolation).

- What is a reasonable profit margin for risk bearing expressed by φ?

- Which risk measure(s) ϱ should be preferred? ($\Rightarrow$ No-acceptability arbitrage!)

- Modeling is often split into different risk modules:

  ⋆ (financial) market risk
  ⋆ insurance risk (underwriting and reserve risks)
  ⋆ credit risk
  ⋆ operational risk

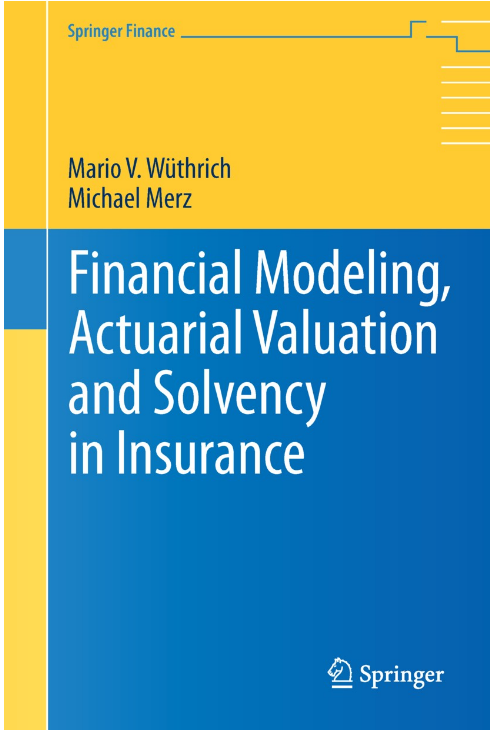
  ▷ Issue: dependence modeling and aggregation of risk modules.

- Aggregation over different accounting years and lines of business?

Dynamic considerations

Are we happy with the above considerations?

▷ **Not entirely!**



Liability run-off is a multi-period problem:

We also want *sensible dynamic behavior*.

This leads to the consideration of multi-period problems and *super-martingales*.