Expected Shortfall revisited

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A few questions to start with

1. **Policyholder perspective:** Assume that for two companies with identical liabilities, policyholders get the same amounts in every state of the world.
   - $\rightarrow$ Is it reasonable that the regulator deems one of these companies to be adequately capitalized but not the other? ($\rightsquigarrow$ *surplus invariance*)
2. **Regulatory arbitrage:** Assume one company can choose from two jurisdictions with different regulators to set up the home office.
   - $\rightarrow$ Is it reasonable that one regulator deems the company adequately capitalized while the other doesn't? ($\rightsquigarrow$ *numéraire invariance*)
3. **Harmonization of global regulation:** Assume regulators agreed globally on a single method for capital requirements in their respective jurisdictions.
   - $\rightarrow$ Would Expected Shortfall (ES) in the relevant local currency be a good choice? What about Value-at-Risk (VaR)?

Objective of the presentation

- The objective of this presentation is to assess how ES performs in terms of surplus and numéraire invariance (formally introduced later)
- Our work complements current debate on ES vs. VaR which focuses exclusively on statistical properties (for an overview of this debate see [3])

Basic question in a capital adequacy framework

Starting point: At time 0 a financial institution selects a portfolio of *assets* and *liabilities* and at time T assets are liquidated and liabilities repaid

- $\rightarrow$ Liability holders worry that the institution may *default* at time T, i.e. that *capital* (= "assets minus liabilities") may become negative at time T, ...
- $\rightarrow$ ... but they are also unwilling to bear the costs of fully eliminating the risk of default and have to settle for some acceptable level of security

Key question for regulators: what is an acceptable level of security for policyholder liabilities, i.e. when should an insurer be deemed to be adequately capitalized?

Testing for capital adequacy: acceptance sets

Capital position of insurers, i.e. assets minus liabilities, at time T are random variables $X : \Omega \to \mathbb{R}$ defined (for simplicity) on a finite state space $\Omega := \{\omega_1, \dots, \omega_n\}$. $\mathscr{X}$ denotes the vector space of all possible capital positions

→ $X(\omega) =$ "value of assets less value of liabilities in state ω"

Regulators subject insurers to a *capital adequacy test* by checking whether their capital positions belong to an *acceptance set* $\mathscr{A} \subset \mathscr{X}$ satisfying two minimal requirements:

→ *Non-triviality:* $\emptyset \neq \mathscr{A} \neq \mathscr{X}$

→ *Monotonicity:* $X \in \mathscr{A}$ and $Y \geq X$ imply $Y \in \mathscr{A}$

Warning: We use interchangeably the terms acceptance set, capital adequacy test, acceptability criterion

The simplest acceptability criterium: scenario testing

The simplest acceptance criterion is testing whether an insurer can meet its obligations on a pre-specified set of states of the world $A \subset \Omega$. The corresponding acceptance sets are called of SPAN-*type* and given by

$$\mathrm{SPAN}(A) := \{X \in \mathscr{X} \; ; X(\omega) \geq 0 \text{ for every } \omega \in A\} \; .$$

- → SPAN stands for Standard Portfolio ANalysis.
- → $\mathrm{SPAN}(A)$ is a closed, coherent acceptance set.
- → In the extreme case $A = \Omega$, the set $\mathrm{SPAN}(A)$ coincides with the set of positive random variables, i.e. an insurer would be required to be able to pay claims in *every* state of the world!

The two most common acceptability criteria: VaR_α and ES_α

The *Value-at-Risk acceptance set* at the level $0 < \alpha < 1$ is the closed, (generally) non-convex cone

$$\mathscr{A}_\alpha := \{X \in \mathscr{X}\,;\ \mathbb{P}(X < 0) \le \alpha\} = \{X \in \mathscr{X}\,;\ \mathrm{VaR}_\alpha(X) \le 0\}\,,$$

where

$$\mathrm{VaR}_\alpha(X) := \inf\{m \in \mathbb{R}\,;\ \mathbb{P}(X + m < 0) \le \alpha\}\ .$$

The *Expected Shortfall acceptance set* at the level $0 < \alpha < 1$ is closed and coherent and defined by

$$\mathscr{A}^\alpha := \{X \in \mathscr{X}\,; \mathrm{ES}_\alpha(X) \le 0\}\ ,$$

where

$$\mathrm{ES}_\alpha(X) := \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)\,d\beta\ .$$

Surplus invariance introduced

Two insurers $X = A - L$ and $Y = A' - L$ with identical liabilities and possibly different assets. Policyholders get the same payments in all states of the world, i.e.

$$D_X = D_Y$$

where $D_X := \max\{-X, 0\}$ and $D_Y := \max\{-Y, 0\}$ are the *options to default*.

Reasonable: X and Y should be either both acceptable or both unacceptable!

Definition ([5])

An acceptance set $\mathscr{A} \subset \mathscr{X}$ is said to be *surplus invariant*, if

$$X \in \mathscr{A},\ Y \in \mathscr{X},\ D_X = D_Y \implies Y \in \mathscr{A}\,.$$

→ We have $X = S_X - D_X$ where $S_X := \max\{X, 0\}$ is the *surplus*. Hence, $\mathscr{A}$ is surplus invariant if acceptability does not depend on the surplus.

Surplus invariance definition: too strong?

Stated in terms of capital positions $X = A - L$ and $Y = A' - L'$ surplus invariance reads

$$A - L \in \mathscr{A}, D_{A-L} = D_{A'-L'} \implies A' - L' \in \mathscr{A}\,.$$

Is this too strong a requirement? Shouldn't we ask this only if $L = L'$?

$$A - L \in \mathscr{A}, D_{A-L} = D_{A'-L} \implies A' - L \in \mathscr{A}\,.$$

No, in fact: both requirements are equivalent!

Numéraire invariance introduced

A two currency world and one insurer with capital position $X_d = A_d - L_d$ expressed in original currency and $X_f = RX_d$ in foreign currency where R is the exchange rate from domestic to foreign. Assume the same test is used in both currencies.

Reasonable: X_d and X_f should be either both acceptable or both unacceptable!

Definition ([4])

An acceptance set $\mathscr{A} \subset \mathscr{X}$ is said to be *numéraire invariant*, if we have

$$X \in \mathscr{A} \text{ and } R \text{ a strictly positive random variable} \implies RX \in \mathscr{A}\,.$$

VaR_α acceptability is surplus and numéraire invariant

Proposition

VaR_α-*acceptability is surplus and* numéraire *invariant*

→ However, this does not invalidate the fundamental criticism of VaR_α:

(a) As long as $\mathbb{P}(X < 0) \leq \alpha$ holds it is blind to what happens on

$$\{\omega \in \Omega \,;\, X(\omega) < 0\}$$

and, therefore, allows the build up of uncontrolled loss peaks on that set!

(b) It does not capture diversification!

Comparing ES- and VaR-acceptability (with easy numbers)

Take $\Omega := \{\omega_1, \ldots, \omega_{10}\}$ with $\mathbb{P}(\omega_1) = \cdots = \mathbb{P}(\omega_{10}) = \frac{1}{10}$. Assume that $X : \Omega \to \mathbb{R}$ is the capital position of an insurance company and, for simplicity, that

$$X(\omega_1) \geq \cdots \geq X(\omega_8) > X(\omega_9) > X(\omega_{10}) .$$

To emulate a "Swiss Solvency Test" type environment we assume an ES test at the confidence level $\alpha = 20\%$, i.e.

$$X \text{ is acceptable} \iff -\underbrace{5}_{\frac{1}{\alpha}} \left[\frac{X(\omega_9)}{10} + \frac{X(\omega_{10})}{10} \right] = \mathrm{ES}_{20\%}(X) \leq 0$$

To emulate a "Solvency II" type environment we assume a VaR test at the higher confidence level $\beta = 15\%$, i.e.

$$X \text{ is acceptable} \iff \mathbb{P}(X < 0) \leq 15\%$$

ES_α acceptability is not surplus invariant

Two companies with capital positions $X := A - L$ and $X' := A' - L$, respectively. They have identical liabilities, different assets, but **identical options to default**, i.e. $D_X = D_{X'}$.

State	L	A	X	D_X	A'	X'	$D_{X'}$
$\omega_1, \ldots, \omega_8$	0	10	10	0	10	10	0
ω_9	12	13	1	0	15	3	0
ω_{10}	12	10	-2	2	10	-2	2

Under the $\mathrm{VaR}_{15\%}$-test we have

$\rightarrow$ $\mathbb{P}(X < 0) = \mathbb{P}(X' < 0) = 10\% \leq 15\% \implies X$ and X' **are both** acceptable

Under $\mathrm{ES}_{20\%}$-test we have

$\rightarrow$ $\mathrm{ES}_{20\%}(X) = -5[\frac{1}{10} - \frac{2}{10}] = \frac{1}{2} > 0 \implies X$ **is not** acceptable

$\rightarrow$ $\mathrm{ES}_{20\%}(X') = -5[\frac{3}{10} - \frac{2}{10}] = -\frac{1}{2} < 0 \implies X'$ **is** acceptable

ES_α acceptability is not numéraire invariant

One a company with capital position $X_d := A_d - L_d$, expressed in domestic currency and $X_f := RX_d$ in foreign currency where R is the exchange rate from domestic to foreign. Assume the domestic and foreign regulators have either both an $\mathrm{ES}_{20\%}$ test or both a $\mathrm{VaR}_{15\%}$ test in their respective currencies.

State	L_d	A_d	X_d	R	X_f
$\omega_1, \dots, \omega_8$	0	10	10	1	10
ω_9	12	13	1	3	3
ω_{10}	12	10	-2	1	-2

Under the $\mathrm{VaR}_{15\%}$-test we have

→ $\mathbb{P}(X_d < 0) = \mathbb{P}(X_f < 0) = 10\% \leq 15\% \implies X_d$ and X_f **are both** acceptable

Under $\mathrm{ES}_{20\%}$-test we have

→ $\mathrm{ES}_{20\%}(X_d) = -5[\frac{1}{10} - \frac{2}{10}] = \frac{1}{2} > 0 \implies X_d$ **is not** acceptable in the domestic jurisdiction

→ $\mathrm{ES}_{20\%}(X_f) = -5[\frac{3}{10} - \frac{2}{10}] = -\frac{1}{2} < 0 \implies X_f$ **is** acceptable in the foreign jurisdiction

ES_α acceptability is <u>not</u> surplus invariant 2

Proposition ([4])

Let $X \notin \mathscr{A}^\alpha$. The following statements are equivalent:

(a) *There exists $Y \in \mathscr{A}^\alpha$ such that $D_X = D_Y$;*

(b) $\mathbb{P}(X < 0) < \alpha$

(c) *$X \in \mathscr{A}_\beta$ for some $\beta \in (0, \alpha)$.*

$\rightarrow$ This situation arises in the region that distinguishes Solvency II (based on $\mathrm{VaR}_{0.5\%}$) and SST (based on $\mathrm{ES}_{1\%}$)

Are coherence, surplus invariance, and numéraire invariance compatible?

Theorem ([4])

Let $\mathscr{A}$ be a closed, **coherent** *acceptance set. The following are equivalent:*

(a) $\mathscr{A}$ *is surplus invariant.*

(b) $\mathscr{A}$ *is numéraire invariant.*

(c) $\mathscr{A}$ *is of* SPAN*-type.*

Corollary ([4])

Let $\mathscr{A}$ be a closed, **convex** *acceptance set. The following are equivalent:*

(a) $\mathscr{A}$ *is numéraire invariant.*

(b) $\mathscr{A}$ *is of* SPAN*-type.*

$\rightarrow$ Unfortunately, unless $A = \Omega$, acceptance sets are of the form $\mathrm{SPAN}(A)$ suffer from a similar shortcoming as VaR_α and are blind to what happens on A^c: They allow the build up of uncontrolled loss peaks on that set!

Conclusion

Multiple competing requirements				
	Captures diversification	Controls loss peaks	Is surplus invariant	Is numéraire invariant
SPAN	✔	✘	✔	✔
VaR	✘	✘	✔	✔
ES	✔	✔	✘	✘

- → This confirms what we all know: **THE** universally ideal capital adequacy test does not exist and we need to weigh the relative importance of competing and, sometimes, mutually exclusive requirements
- → Expected Shortfall does not really take an exclusive policyholder perspective
- → A global Expected Shortfall regime would allow for regulatory arbitrage
- → The SPAN-type acceptance sets are the only coherent acceptance sets that are surplus invariant and also the only ones that are numéraire invariant

THANK YOU FOR YOUR ATTENTION!

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