Approximating irregular SDEs via iterative Skorokhod embeddings

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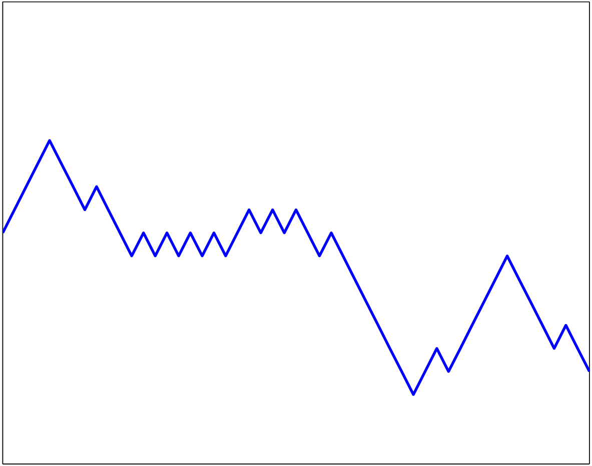
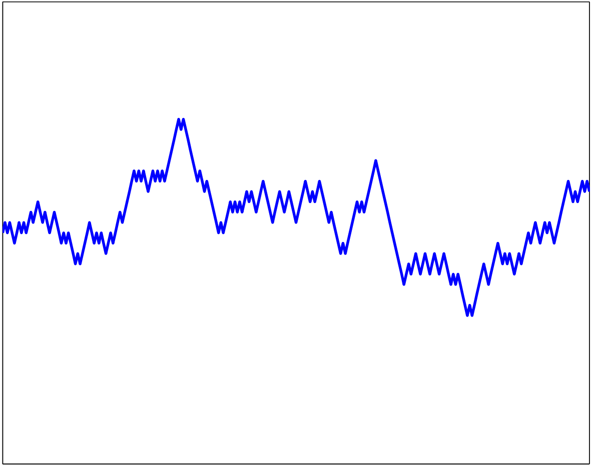
Outline

Outline

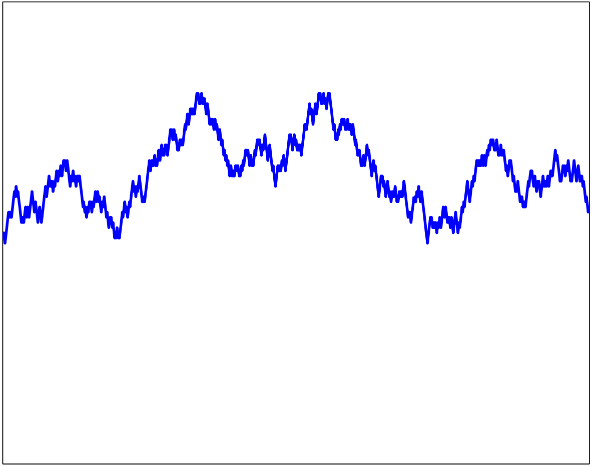
Random walk



Shrink time N times and space $\sqrt{N}$ times



Limit as $N \to \infty$



The Donsker–Prokhorov invariance principle

Input: $\mu \neq \delta_0$ centered probability measure on $\mathbb{R}$ with $\int x^2\,\mu(dx) < \infty$

Theorem

Let $X_1, X_2, \dots$ be iid random variables with distribution μ, let $Y_0^N = 0$ and

$$Y_k^N = Y_{k-1}^N + \sqrt{\frac{1}{N}}X_k.$$

Set

$$Y_t^N = Y_{\lfloor t\rfloor}^N + (t - \lfloor t\rfloor)(Y_{\lfloor t\rfloor+1}^N - Y_{\lfloor t\rfloor}^N).$$

Then $\left(Y_{Nt}^N\right)_{t\in\mathbb{R}_+}$ converges in distribution to a Brownian motion times $\sqrt{\int x^2\,\mu(dx)}$ as $N \to \infty$.

Our setting: input

- $\mu \neq \delta_0$ centered probability measure on $\mathbb{R}$
- 1-dim driftless diffusion M with state space $I \subseteq \mathbb{R}$

$$dM_t = \eta(M_t)\,dW_t, \quad M_0 = m \in I,$$

$\eta\colon I \to \mathbb{R}$ may be irregular

Main question

$\mu \neq \delta_0$ centered probability measure on $\mathbb{R}$
$dM_t = \eta(M_t)\, dW_t,\ M_0 = m$

Can we choose a scale factor $a_N \colon I \to (0, \infty)$ such that we have:

Theorem

Let $X_1, X_2, \ldots$ be iid random variables with distribution μ, let $Y_0^N = m$ and

$$Y_k^N = Y_{k-1}^N + a_N(Y_{k-1}^N) X_k.$$

Set

$$Y_t^N = Y_{\lfloor t \rfloor}^N + (t - \lfloor t \rfloor)(Y_{\lfloor t \rfloor + 1}^N - Y_{\lfloor t \rfloor}^N).$$

Then $(Y_{Nt}^N)_{t \in \mathbb{R}_+}$ converges in distribution to M as $N \to \infty$.

Answer: yes

$$dM_t = \eta(M_t)\,dW_t,\ \ M_0 = m$$
$$\mu \neq \delta_0 \text{ centered probability measure on } \mathbb{R}$$
$$Y_k^N = Y_{k-1}^N + a_N(Y_{k-1}^N)X_k$$
$$X_k \text{ iid} \sim \mu$$

- We will construct scale factors a_N via iterative Skorokhod embeddings of shifted and scaled μ into M
  $\rightsquigarrow$ Remaining question: how to determine a_N?
- This method works even for irregular or quickly growing μ
- We will compare our scheme with the "weak Euler scheme" that corresponds to scale factors

$$a_N^{Eu}(y) = \frac{\eta(y)}{\sqrt{N \int x^2\,\mu(dx)}}$$

$\rightsquigarrow$ Examples: $\mu = N(0,1)$, $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$

Comparison with the Euler scheme, I

The Euler scheme [determined by $y \mapsto a_N^{Eu}(y)$]

- η globally Lipschitz (in particular, of linear growth) $\rightsquigarrow$ the Euler scheme works good
- Several papers on the Euler scheme with irregular drift but regular diffusion coefficient (Gyöngy, Krylov, ...)
- [Yan 2002]: Irregular diffusion coefficient η (still Leb-a.e. continuous) but of linear growth
  $\rightsquigarrow$ Tanaka's example: $\eta(y) = 1_{\{y>0\}} - 1_{\{y\le 0\}}$
  - $(Y_{Nt}^{N,Eu})$ converges in distribution to M
  - Rates only under Hölder continuity of η

Our scheme [determined by $y \mapsto a_N(y)$]

- (Y_{Nt}^N) converges in distribution to M under less regularity and growth assumptions (η Borel measurable and locally bounded away from 0 and $\pm\infty$)
- Order of convergence $1/4$ regardless of regularity of η (η bounded away from 0 and $\pm\infty$)
- But scale factors a_N are more difficult to find than a_N^{Eu}

Comparison with the Euler scheme, II

Plus examples showing that lack of

- regularity and/or
- linear growth

can indeed make the Euler scheme diverge

([Hutzenthaler, Jentzen, Kloeden 2010] and below)

Recall Skorokhod's proof of the Donsker–Prokhorov invariance principle

$\mu \neq \delta_0$ centered probability measure on $\mathbb{R}$ with $\int x^2\,\mu(dx) = 1$

X_k iid $\sim \mu, \quad Y_k^N = Y_{k-1}^N + \sqrt{\frac{1}{N}} X_k$

Skorokhod embeds Y^N into a BM B with stopping times $0 = \tau^N(0) < \tau^N(1) < \cdots$, i.e.

$$(B_{\tau^N(k)}; k \geq 0) \stackrel{d}{=} (Y_k^N; k \geq 0),$$

where $\tau^N(k) - \tau^N(k-1)$ are iid with $\mathsf{E}[\tau^N(k) - \tau^N(k-1)] < \infty$. Then, by Wald's identity,

$$\mathsf{E}[\tau^N(k) - \tau^N(k-1)] = \mathsf{E}(B_{\tau^N(k)} - B_{\tau^N(k-1)})^2 = \mathsf{E}\left(\sqrt{\frac{1}{N}} X_k\right)^2 = \frac{1}{N}.$$

One can show that $(B_{\tau^N(k)})_{k\geq 0}$ converges to B in probability, hence $(Y_k^N)_{k\geq 0}$ converges to B in distribution.
$\rightsquigarrow$ Clue to "remaining question": follow Skorokhod's approach

Outline

Idea

Details

Examples

The Skorokhod embedding problem for M

Weak solution of the SDE

Interior of the state space:

- $I = (l, r)$ with $-\infty \leq l < r \leq \infty$

SDE

$$dM_t = \eta(M_t)\, dW_t, \quad M_0 = y \in I$$

$\eta\colon I \to \mathbb{R}$ Borel measurable with

$$\eta(x) \neq 0 \text{ for all } x \in I$$
$$\frac{1}{\eta^2} \in L^1_{\text{loc}}(I)$$
$$\eta(x) := 0 \text{ for all } x \in \mathbb{R} \setminus I$$

Theorem (Engelbert & Schmidt 1985)

There is a weak solution of the SDE and we have uniqueness in law.

The Skorokhod embedding problem for M

Let $(M_t)_{t\geq 0}$ be a solution to

$$dM_t = \eta(M_t)\,dW_t, \quad M_0 = y \in I$$

SEP: Given a distribution ν with $\int x\,\nu(dx) = y$, find a stopping time τ (if any) such that

$$M_\tau \sim \nu$$

and express $\mathsf{E}\tau$ in terms of η and ν

Integrable solutions of the SEP for M, idea

$$dM_t = \eta(M_t)\, dW_t,\ M_0 = y$$
$$\textstyle\int x\,\nu(dx) = y$$

Let

$$q(y,x) = \int_y^x \int_y^u \frac{2}{\eta^2(z)}\, dz\, du, \quad y \in I,\ x \in \mathbb{R}.$$

By Itô's formula,

$$q(y, M_t) - t, \quad t \ge 0,$$

is a local martingale starting from 0. If it is a true martingale, and the optional sampling theorem applies for a solution τ of the SEP for M, then

$$\mathsf{E}\tau = \mathsf{E}q(y, M_\tau) = \int q(y,x)\,\nu(dx).$$

- Role of the argument y of q: starting point of M
- Formally, the latter integral is the minimal possible $\mathsf{E}\tau$

Integrable solutions of the SEP for M, result

$$dM_t = \eta(M_t)\,dW_t,\ \ M_0 = y$$
$$\int x\,\nu(dx) = y$$

Theorem (Ankirchner, Hobson, Strack 2013)

If $\int q(y,x)\,\nu(dx) < \infty$, then there exists a stopping time τ such that $M_\tau \sim \nu$. Moreover, we can choose τ such that

$$E[\tau] = \int q(y,x)\,\nu(dx).$$

Proposition (Ankirchner, Hobson, Strack 2013)

Any stopping time that solves the SEP for M satisfies $E[\tau] \geq \int q(y,x)\,\nu(dx)$.

Further remarks to $q(y, x)$, $y \in I$, $x \in \mathbb{R}$

$$q(y,x) = \int_y^x \int_y^u \frac{2}{\eta^2(z)}\, dz\, du, \quad y \in I,\ x \in \mathbb{R}$$

Regardless of the value $y \in I$ it holds:

- On I, $x \mapsto q(y, x)$ is strictly convex, nonnegative, strictly decreasing to zero on $(l, y]$, strictly increasing from zero on $[y, r)$
- $q(y, x) = \infty$ for $x \in \mathbb{R} \setminus [l, r]$
- q is the function from Feller's test for explosions:

$$l \text{ (resp. } r\text{) is accessible} \iff q(y,l) < \infty \quad (\text{resp. } q(y,r) < \infty)$$

- If $l = -\infty$ (resp. $r = \infty$), then $q(y, l) = \infty$ (resp. $q(y, r) = \infty$)

Consecutive embeddings

Embedding Markov chains

- $dM_t = \eta(M_t)\,dW_t$, $M_0 = m$
- $\mu \neq \delta_0$ centered probability measure and $X_k \sim \mu$
- $N \in \mathbb{N}$ is fixed

Problem (P). Does there exist
1. a measurable function $a_N \colon I \to (0, \infty)$ (scale factor) and
2. a sequence of $(\mathcal{F}_t)$-stopping times $(\tau_k^N)_{k\in\mathbb{Z}_+}$ with $\tau_0^N = 0$ satisfying

$$E[\tau_{k+1}^N - \tau_k^N | \mathcal{F}_{\tau_k^N}] = \frac{1}{N}$$

such that

$$(M_{\tau_k^N})_{k\in\mathbb{Z}_+} \stackrel{d}{=} (Y_k^N)_{k\in\mathbb{Z}_+},$$

where

$$Y_0^N = m, \quad Y_k^N = Y_{k-1}^N + a_N(Y_{k-1}^N)X_k$$

Embed the transition probabilities consecutively

Given that τ_k^N is already constructed and that $M_{\tau_k^N} = y$, embed the distribution $\mu\left(\frac{\cdot - y}{a_N(y)}\right)$ into $(M_{\tau_k^N + t})$, where the embedding time $\rho_k^N(y)$ satisfies

$$E[\rho_k^N(y)|\mathcal{F}_{\tau_k^N}] = \frac{1}{N}. \qquad (*)$$

Then define $\tau_{k+1}^N = \tau_k^N + \rho_k^N(M_{\tau_k^N})$.

Determining scale factor (red off from formula $(*)$):
For each $y \in I$ find a solution $a_N(y) \in (0, \infty)$ of the equation

$$\int_{\mathbb{R}} q(y, x)\, \mu\left(\frac{dx - y}{a_N(y)}\right) = \frac{1}{N}.$$

$\rightsquigarrow$ answer to "remaining question"

- Difficult, but many explicit examples

"Remaining question" $\rightsquigarrow$ Problem (P)

For each $y \in I$ define

$$G_y(a) := \int_{\mathbb{R}} q(y,x)\,\mu\left(\frac{dx-y}{a}\right) = \int_{\mathbb{R}} q(y,ax+y)\,\mu(dx), \quad a \geq 0$$

Recall: $G_y(a)$ minimal expected time needed for embedding $\mu\left(\frac{\cdot - y}{a}\right)$ into M conditionally to $M_0 = y$.

If for all $y \in I$ there is a solution $a_N(y)$ to $G_y(a) = \frac{1}{N}$, then there is a solution to Problem (P).

Question: Does $\frac{1}{N}$ always belong to the image of $G_y : [0,\infty) \to [0,\infty]$?

- G_y strictly increasing, left-continuous, but can jump to ∞

Answer: In general, no.

Summary to Problem (P), I

Messages:

- In many cases there exists a solution of Problem (P).
  We have sufficient conditions in terms of η and μ.
- Some μ always work. For instance, $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$, or μ with a compact support satisfying

$$\mu(\{\inf \operatorname{supp} \mu\}) > 0 \quad \text{and} \quad \mu(\{\sup \operatorname{supp} \mu\}) > 0.$$

- For some η no restrictions on μ (except for some “minimal natural restrictions”).
- Example of a “minimal natural restriction” on μ:
  if $l > -\infty$, then $\inf \operatorname{supp} \mu > -\infty$.

Summary to Problem (P), II

Given η, how restrictive are our assumptions on μ guaranteeing the solvability of Problem (P)?

Look at examples:

- Brownian motion [$\eta \equiv 1$, $I = \mathbb{R}$]:
  $\eta \neq \delta_0$ centered distribution with $\int x^2 \, \mu(dx) < \infty$
  $\rightsquigarrow$ exactly as in the Donsker-Prokhorov invariance principle
- Absorbed at zero Brownian motion [$\eta \equiv 1$ on $I = (0, \infty)$]:
  $\eta \neq \delta_0$ centered distribution with $\int x^2 \, \mu(dx) < \infty$,
  $\inf \operatorname{supp} \mu > -\infty$, and $\mu(\{\inf \operatorname{supp} \mu\}) > 0$
- Geometric Brownian motion [$\eta(x) = x$ on $I = (0, \infty)$]:
  $\eta \neq \delta_0$ centered distribution with $\inf \operatorname{supp} \mu > -\infty$

Summary to Problem (P), III

A somewhat puzzling message (given the discussion above):

There are situations when $\frac{1}{N}$ does not belong to the image of G_y, but Problem (P) has a solution.

In particular,

$$\mathsf{E}[\tau_{k+1}^N - \tau_k^N | \mathcal{F}_{\tau_k^N}] = \frac{1}{N} \quad \text{a.s.},$$

but

$$\mathsf{P}\left(\mathsf{E}[G_{M_{\tau_k^N}}(a_N(M_{\tau_k^N}))] \neq \frac{1}{N}\right) > 0.$$

$\rightsquigarrow$ Example: absorbed at zero Brownian motion

Some properties of scale factor a_N (to be compared with the Euler scheme)

Asymptotic behavior of the scale factors

$$a_N(y) \text{ vs. } a_N^{Eu}(y) = \eta(y)/\sqrt{N}$$
$$Y_k^N = Y_{k-1}^N + a_N(Y_{k-1}^N)X_k$$

For simplicity assume that $\int x^2 \, \mu(dx) = 1$ and let η^* denote the upper semicontinuous envelope of $|\eta|$.

Theorem

(i) For any $y \in I$, we have

$$\limsup_{N\to\infty} \sqrt{N} a_N(y) \le \eta^*(y)$$

(ii) If $1/|\eta|$ is bounded and η is continuous at $y \in I$, then

$$\lim_{N\to\infty} \sqrt{N} a_N(y) = |\eta(y)|.$$

- On one hand, our scale factors a_N are similar to a_N^{Eu}

Lip(1) and linear growth

Proposition

Assume $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$ and $I = \mathbb{R}$. Then all mappings $y \mapsto a_N(y)$ are Lip(1). In particular, they have linear growth with a (unit) slope not depending on N.

- On the other hand, our scale factors a_N are different from $a_N^{Eu}(y) = \eta(y)/\sqrt{N}$
  - smoothing if η has irregularities
  - tempered growth behavior

Comparison Principle

- Let $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$ and $X_k \sim \mu$
- Conditionally on $\{Y_k = y\}$ we have

$$Y_{k+1}^N = Y_{k+1}^N(y) = y + a_N(y)X_{k+1} = y \pm a_N(y).$$

- **Question:** Do we have $Y_{k+1}^N(y) \geq Y_{k+1}^N(y')$ if $y \geq y'$?

Proposition

The mappings $y \mapsto y \pm a_N(y)$ are nondecreasing.

- The Euler Scheme $a_N^{Eu}(y) = \eta(y)/\sqrt{N}$

$$Y_{k+1}^{N,Eu}(y) = y + a_N^{Eu}(y)X_{k+1} = y \pm a_N^{Eu}(y)$$

does in general not satisfy a comparison principle.

Convergence in distribution

Setting for convergence in distribution

- Assume sufficient conditions for solvability of Problem (P)
- $(Y_k^N, \tau^N(k), k \in \mathbb{Z}_+)$ solution of Problem (P), in particular

$$(Y_k^N; k \geq 0) \stackrel{d}{=} (M_{\tau^N(k)}; k \geq 0)$$

- Extension to continuous time:

$$Y_t^N = Y_{\lfloor t \rfloor}^N + (t - \lfloor t \rfloor)(Y_{\lfloor t \rfloor + 1}^N - Y_{\lfloor t \rfloor}^N)$$

Question: When does it hold that the processes (Y_{Nt}^N) converge to (M_t) in distribution?

Convergence results

(C1) $|\eta|$ and $\frac{1}{|\eta|}$ are bounded on I.

Theorem

Assume (C1). Then the processes (Y^N_{Nt}) converge to (M_t) in distribution.

(C2) $|\eta|$ and $\frac{1}{|\eta|}$ are locally bounded on I.

Theorem

Suppose (C2) and that μ has a compact support. Then the processes (Y^N_{Nt}) converge to (M_t) in distribution.

Convergence rate

- Under (C1) the order of convergence is $1/4$
  (η is just Borel measurable)

- Under (C2) this is no longer true
  (a counterexample)

Comparison with the Euler scheme

To the best of our knowledge, only [Yan 2002] treats the Euler scheme with irregular diffusion coefficient, but, as for convergence rate, η should be Hölder continuous there

Outline

Idea

Details

Examples

Summary of the algorithm

Aim: Approximate distributional properties of M, $dM_t = \eta(M_t)\, dW_t$, $M_0 = m$.

1. Determine $q(y, x) = \int_y^x \int_y^u \frac{2}{\eta^2(z)} dzdu$.
2. Choose the number of time steps $N \in \mathbb{N}$.
3. Choose a reference measure μ such that Problem (P) has a solution.
4. Solve in a the equation $\int_{\mathbb{R}} q(y, ax + y)\, \mu(dx) = 1/N$ for all $y \in I$ $\rightsquigarrow$ solution $a_N(y)$.
5. Simulate $Y_k^N = Y_{k-1}^N + a_N(Y_k^N) X_k$, $Y_0 = m$, where X_k iid $\sim \mu$.

Overview of explicit examples

1. Brownian motion
2. Geometric Brownian motion
   - Both our and the Euler scheme work
3. Brownian motion absorbed at zero
4. Diffusion between two media

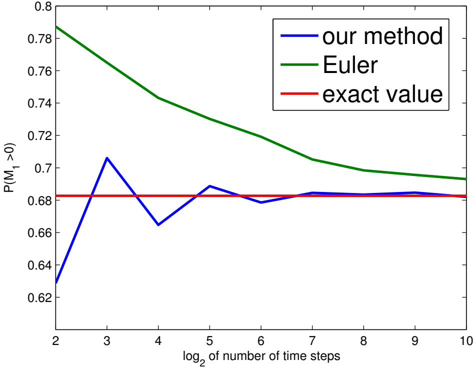
$$I = \mathbb{R}, \quad \eta = 1_{(0,\infty)} + A 1_{(-\infty,0]}$$

   - In both schemes convergence holds, we have convergence order $1/4$, in the Euler scheme order unknown (?)

An illustration for a Brownian motion absorbed at zero

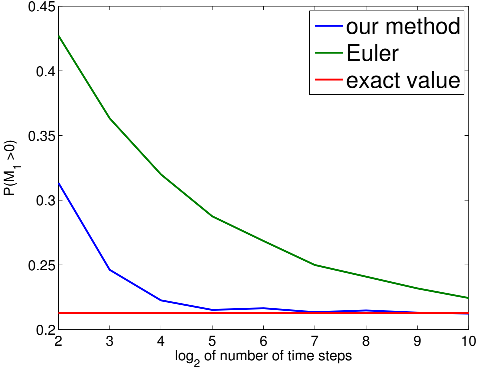
$dM_t = 1_{\{M_t>0\}} dW_t$

- $M_0 = 1$
- Estimate $P(M_1 > 0)$



$dM_t = 1_{\{M_t>0\}} dW_t$

- $M_0 = 0.27$
- Estimate $P(M_1 > 0)$



Two final explicit examples

- $I = \mathbb{R}$, $\eta(x) = \cosh(x)$
  $\rightsquigarrow$ Exponentially growing η
  $\rightsquigarrow$ The Euler scheme diverges, our scheme converges, but we do not have rates
- $I = \mathbb{R}$, $\eta(x) = \frac{1}{|x|}$, $\eta(0) = 1$
  $\rightsquigarrow$ Not locally bounded η
  $\rightsquigarrow$ The Euler scheme diverges, our convergence results do not apply, but we can show that our scheme converges (all is explicit, this helps), but we do not have rates

Exponentially growing η

$$I = \mathbb{R}, \quad \eta(x) = \cosh(x)$$

Exponentially growing η

- $dM_t = \cosh(M_t)dW_t$, $M_0 = 0$.
- $q(y,x) = 2\left[\log\left(\frac{\cosh(x)}{\cosh(y)}\right) - \tanh(y)(x-y)\right]$, for $y, x \in \mathbb{R}$.
- Choose $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$
- Then for $N \in \mathbb{N}$ and $y \in \mathbb{R}$

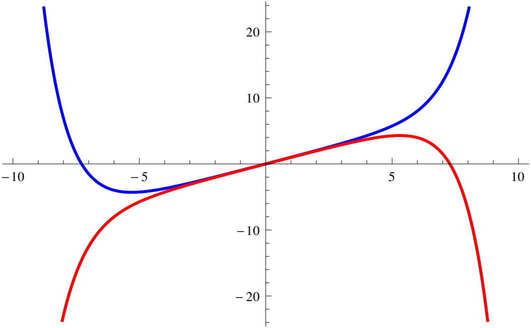
$$a_N(y) = \frac{1}{2}\operatorname{arcosh}\left(2(\exp(1/N) - 1)\cosh^2(y) + 1\right).$$

- Euler scheme

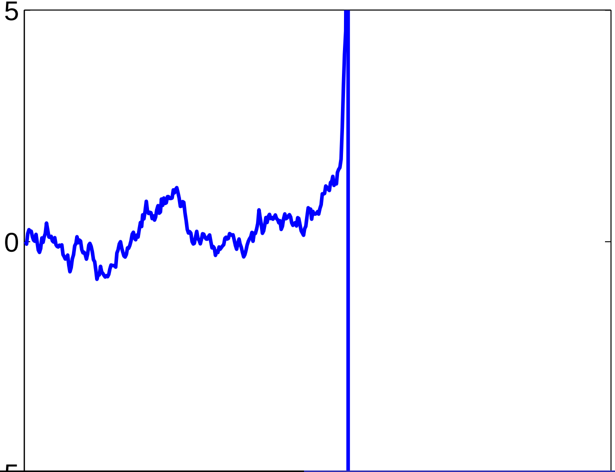
$$a_N^{Eu}(y) = \frac{1}{\sqrt{N}}\cosh(y)$$

Scale factor for the Euler scheme $\rightsquigarrow$ "saw" effect

- $N = 10000$
- $y \mapsto y + a_N^{Eu}(y)$ and $y \mapsto y - a_N^{Eu}(y)$

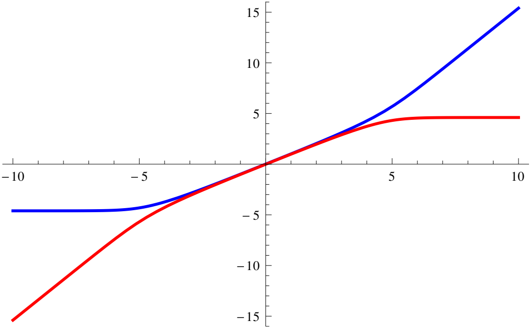


A realization of the Euler scheme for $dM_t = \cosh(M_t)dW_t$: “saw” effect

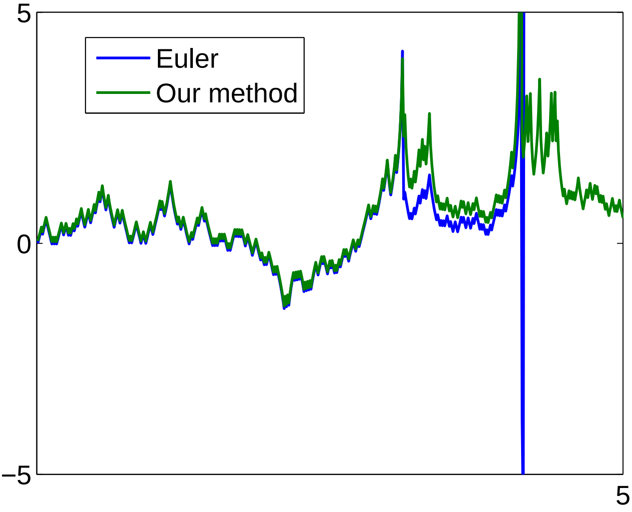


Scale factor for our scheme: the “saw” is not possible due to the comparison principle

- $N = 10000$
- $y \mapsto y + a_N(y)$ and $y \mapsto y - a_N(y)$



Comparing realizations for $dM_t = \cosh(M_t)dW_t$



Approximating an expectation of M_1

Aim: For $\alpha \in (0,1)$ approximate $\mathrm{E}[|M_1|^\alpha]$ numerically.

- Euler scheme $Y^{N,Eu}$ converges a.s. to M ([Gyöngy 1998], for Gaussian increments).
- But it follows from [Hutzenthaler, Jentzen, Kloeden 2010] that $\mathrm{E}[|Y_N^{N,Eu}|^\alpha] \to \infty$ as $N \to \infty$.

Proposition

The family $(|Y_N^N|^\alpha)_{N\in\mathbb{N}}$ is uniformly integrable. Hence,

$$E[|Y_N^N|^\alpha] \to E[|M_1|^\alpha]$$

as $N \to \infty$.

Not locally bounded η

$$I = \mathbb{R}, \quad \eta(x) = \frac{1}{|x|}, \quad \eta(0) = 1$$

Not locally bounded η

- $dM_t = \eta(M_t)dW_t$, $M_0 = 0$.
- $\eta(x) = \frac{1}{|x|}$ for $x \neq 0$ and $\eta(0) = 1$.
- $q(y,x) = \frac{1}{6}x^4 - \frac{2}{3}xy^3 + \frac{1}{2}y^4$, for $y, x \in \mathbb{R}$.
- Choose $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$
- Then for $N \in \mathbb{N}$ and $y \in \mathbb{R}$

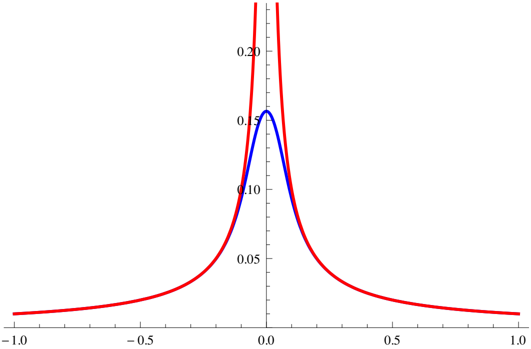
$$a_N(y) = \sqrt{\sqrt{9y^4 + \frac{6}{N}} - 3y^2}.$$

- Euler scheme

$$\tilde{a}_N(y) = \frac{1}{\sqrt{N}} \frac{1}{|y|} \text{ for } y \neq 0, \quad \tilde{a}_N(0) = \frac{1}{\sqrt{N}}.$$

Scale factors

- $N = 10000$, $y \mapsto a_N(y)$ and $y \mapsto \tilde{a}_N(y)$



- $a_N(0) = \sqrt[4]{\frac{6}{N}}$ and $\lim_{N\to\infty} \sqrt{N} a_N(y) = \frac{1}{|y|}$.

Convergence

- The Euler approximation $\tilde{Y}^N$ does not converge in distribution to M: For every $N \in \mathbb{N}$ we have

$$\tilde{Y}_2^N = \frac{X_1}{\sqrt{N}} + \frac{X_2}{X_1}.$$

Proposition

The sequence of continuous processes $(Y_{Nt}^N)_{t \geq 0}$ converges in law to the process M, as $N \to \infty$. Moreover, we have

$$E[f(Y_N^N)] \to E[f(M_1)]$$

as $N \to \infty$ for every continuous function $f : \mathbb{R} \to \mathbb{R}$ with $|f(x)| \leq c(1 + |x|^\alpha)$, $x \in \mathbb{R}$, for some $c \in \mathbb{R}_+$ and $\alpha \in (0, 4)$.

Conclusion

- We constructed Markov chains that can be embedded into a driftless diffusion with a **fixed mean time lag** $\frac{1}{N}$ ...
- ...and a **"non-local", "implicit"** numerical scheme to approximate diffusions with irregular coefficients and superlinear growth
- The scale factors may differ significantly from their counterparts in the Euler scheme
  - **smoothing** if η has discontinuities
  - **tempered** growth behavior

Thank you!