(No-)betting Pareto-optima under rank-dependent utility

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joint work with Mario Ghossoub (University of Waterloo), and an old version of the paper can be downloaded from SSRN:
http://papers.ssrn.com/sol3/papers.cfm?abstract_id=3524926.
A new version is available upon request.

Risk measures and uncertainty in insurance, Hannover,

May 20th, 2022

University of Amsterdam

To Bet or Not to Bet... In EUT

- When is it Pareto-optimal for risk-averse agents to take bets?

  $\Longrightarrow$ Starting from an environment with no aggregate uncertainty, under what conditions is it Pareto-improving to introduce uncertainty through betting (trade of an uncertain asset)?

- One obvious case is when the agents are risk-averse EU-maximizers and do not share beliefs (Billot et al., 2000, ECMA):

  $\Longrightarrow$ If the agents disagree on probability assessments, then they find it Pareto-improving to engage in uncertainty-generating trade (i.e., to bet):

$$\text{Disagreement about beliefs} \overset{EUT}{\Longrightarrow} \text{Betting is Pareto-improving}$$

  $\Longrightarrow$ Conversely, disagreement about probabilities is the only way that betting may be Pareto-improving when starting from a no-betting allocation:

$$\text{Common beliefs} \overset{EUT}{\Longrightarrow} \text{Betting is not Pareto-improving (no-betting PO)}$$

Bilateral Risk Sharing: The Main Idea

- We examine a situation in which both the agent and the counterparty are Rank-Dependent Utilities (RDU), with different probability weighting functions of the same underlying probability measure.

- RDU is popular to model *ambiguity*-aversion and the over-weighing of the probabilities of extreme (good and/or bad) events.

$\Longrightarrow$ We show that, as long as the agents' probability weighting functions satisfy a certain consistency requirement, PO allocations are no-betting allocations.
$\Longrightarrow$ For instance, when both agents are risk-averse (in a weak sense).

$\Longrightarrow$ Otherwise, betting is PO.

Setting

- Let (S, Σ, P) be a non-atomic probability space, and let $L^1(S, \Sigma, P)$ be the space of all integrable, $\mathbb{R}$-valued, and Σ-measurable functions on (S, Σ, P).
- There are two agents who seek a betting arrangement.
- We assume no aggregate uncertainty in this economy, and the aggregate wealth is given by $W \in \mathbb{R}$.
- A (feasible) *allocation* is a pair $(X_1, X_2) \in L^1(S, \Sigma, P) \times L^1(S, \Sigma, P)$ such that $X_1 + X_2 = W$.

$\Longrightarrow$ **Trading is therefore seen as betting rather than as hedging or risk-sharing**.

Setting

- A feasible allocation (X_1, X_2) is a Pareto-improvement over another feasible allocation (Y_1, Y_2) if
$$U_i(X_i) \geqslant U_i(Y_i),$$
for $i \in \{1, 2\}$, with at least one strict inequality.

- A feasible allocation (X_1^*, X_2^*) is Pareto-Optimal (PO) if there is no other feasible allocation $(\tilde{X}_1, \tilde{X}_2)$ that is a Pareto-improvement over (X_1^*, X_2^*).

- A feasible allocation (X_1, X_2) is a no-betting allocation if for some $i \in \{1, 2\}$ and some constant $c \in \mathbb{R}$, $X_i = c$, P-a.s. (and hence $X_{3-i} = W - c$, P-a.s.)

$\implies$ For example, $(\alpha W, (1-\alpha) W)$ is a no-betting allocation, for any $\alpha \in \mathbb{R}$.

Assumptions

- The preferences of Agent 1 are to maximize:

$$U_1(Z) = \int u_1(Z)\ dT_1 \circ P := \int_0^{+\infty} T_1\left(P\left(\{s \in S : u_1(Z(s)) > t\}\right)\right)\ dt$$
$$+ \int_{-\infty}^{0} \left[T_1\left(P\left(\{s \in S : u_1(Z(s)) > t\}\right)\right) - 1\right]\ dt = \int u_1(x) T_1'(1 - F_Z(x)) dF_Z(x).$$

- The preferences of Agent 2 are to maximize:

$$U_2(Z) = \int u_2\Big(Z\Big)\ dT_2 \circ P.$$

- The utility functions u_i are increasing, strictly concave, continuously differentiable, and satisfy the Inada conditions $\lim_{x \to -\infty} u_i'(x) = +\infty$ and $\lim_{x \to +\infty} u_i'(x) = 0$.
- The probability weighting functions $T_i : [0,1] \to [0,1]$ are such that $T_i(0) = 0$, $T_i(1) = 1$, and functions T_i are absolutely continuous and increasing.

What if there is aggregate uncertainty?

- In part, an open question...
- Chateauneuf et al. (2000), Carlier and Dana (2008), Chakravarty and Kelsey (2015) all assume that the probability weighting functions are convex.
- Xia and Zhou (2016) assume that all agents use the same probability weighting function.
- Jin et al. (2019) show that Pareto optimal risk-sharing contracts exist under technical conditions that require aggregate market uncertainty.
- It is well-known in economics that (no) aggregate uncertainty matters (Billot et al., 2000, 2002; Chateauneuf et al., 2000; Ghirardato and Siniscalchi, 2018; **B** and Ghossoub, 2020).

Setting

$$\left(\widehat{\mathcal{P}}_{V_0}\right) \qquad \sup_{Y \in L^1(S,\Sigma,P)} \left\{ \int u_1\left(W - Y\right)\, dT_1 \circ P : \int u_2\left(Y\right)\, dT_2 \circ P \geqslant V_0 \right\}.$$

Lemma

(i) *If the allocation (X_1^*, X_2^*) is PO, then X_2^* solves Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$ with $V_0 := U_2\left(X_2^*\right)$.*

(ii) *For a given $V_0 \in \mathbb{R}$, any solution Y^* to Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$ leads to an allocation $\left(W - Y^*, Y^*\right)$ that is PO.*

(iii) *If $Y^* \in L^1\left(S, \Sigma, P\right)$ solves Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$ for a given $V_0 \in \mathbb{R}$, then $U_2\left(Y^*\right) = V_0$.*

Optimal Betting Between Two RDU Agents

Theorem

A feasible allocation (X_1^*, X_2^*) is Pareto-Optimal if there exists some $\lambda^* > 0$ such that

$$X_2^* = m^{-1}\left(\lambda^* \delta'\left(T_1(U)\right)\right), \text{ where:}$$

- *U is a random variable on (S, Σ, P) with a uniform distribution on $(0,1)$;*
- *$m(x) := \frac{u_1'(W-x)}{u_2'(x)}$, for all $x \geqslant 0$;*
- *δ is the convex envelope on $[0,1]$ of the function $\Psi : [0,1] \to \mathbb{R}$ defined by $\Psi(t) := \widetilde{T}_2\left(T_1^{-1}(t)\right)$, where $\widetilde{T}_2(t) = 1 - T_2(1-t)$, for each $t \in [0,1]$.*

Moreover, for every PO allocation (X_1^*, X_2^*), there exists a $\lambda^* > 0$ such that X_2^* has the same distribution as $m^{-1}\left(\lambda^* \delta'\left(T_1(U)\right)\right)$ under P.

Convex envelope

- The convex envelope of a function is the largest convex function that is point-wise dominated by that function
  - Thus, the convex envelope of f on the interval $[0,1]$ is defined as the greatest convex function g on $[0,1]$ such that $g(x) \leqslant f(x)$, for each $x \in [0,1]$.

Corollary

If both T_1 and T_2 are concave, then a feasible allocation (X_1^*, X_2^*) is PO if there exists some $\lambda^* > 0$ such that

$$X_2^* = m^{-1}\left(\lambda^*\left(\frac{T_2'(1-U)}{T_1'(U)}\right)\right).$$

Moreover, for every PO allocation (X_1^*, X_2^*), there exists a $\lambda^* > 0$ such that X_2^* has the same distribution as $m^{-1}\left(\lambda^*\left(\frac{T_2'(1-U)}{T_1'(U)}\right)\right)$ under P.

Example with Inverse S-shaped Probability Weighting Functions

- As in Tversky and Kahneman (1992), let the probability weighting function T_i be given by:

$$T_i\left(t\right) = \frac{t^{\gamma_i}}{\left(t^{\gamma_i} + \left(1-t\right)^{\gamma_i}\right)^{1/\gamma_i}}, \ \forall t \in [0,1],$$

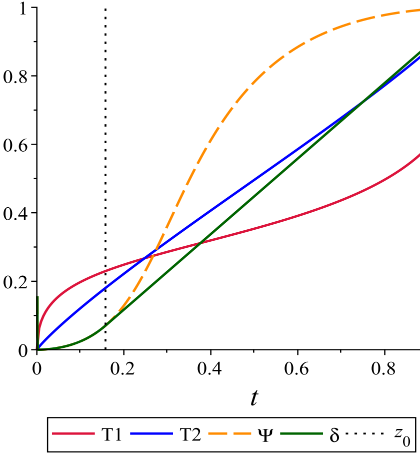
  for some $\gamma_i \in (0,1]$.

- It then follows that:

$$\Psi\left(t\right) = 1 - \frac{\left(1 - T_1^{-1}\left(t\right)\right)^{\gamma_2}}{\left(\left(T_1^{-1}\left(t\right)\right)^{\gamma_2} + \left(1 - T_1^{-1}\left(t\right)\right)^{\gamma_2}\right)^{1/\gamma_2}}, \ \forall t \in [0,1].$$

Example, continued

Let $\gamma_1 = 0.5$ and $\gamma_2 = 0.9$. Then:



Example, continued

- Let $W = 0$ and $u_i(x) = \frac{-\exp(-\beta_i x)}{\beta_i}$, for $x \in \mathbb{R}$ and $\beta_i > 0$.
- $m(x) = \exp((\beta_1 + \beta_2)x)$ for $x \in \mathbb{R}$, and so $m^{-1}(y) = \ln(y)/(\beta_1 + \beta_2)$ for $y > 0$.
- Let $\beta_1 = 0.5$ and $\beta_2 = 0.5$. A feasible allocation (X_1^*, X_2^*) is PO if there exists some $\lambda^* > 0$ such that

$$X_2^* = m^{-1}\left(\lambda^* \delta'\left(T_1(U)\right)\right) = \left(\frac{1}{\beta_1 + \beta_2}\right) \ln\left(\lambda^* \delta'\left(T_1(U)\right)\right)$$
$$= \ln(\lambda^*) + \ln\left(\delta'\left(T_1(U)\right)\right).$$

- Thus, the choice of $\lambda^* > 0$ leads to a deterministic side-payment (positive or negative), in addition to the betting contract $I^*(U) := \ln\left(\delta'\left(T_1(U)\right)\right)$.

Example, continued

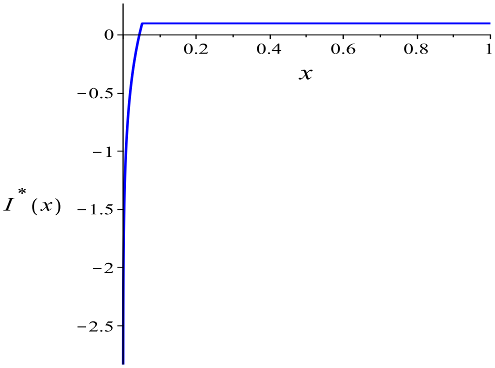


Figure: This graph plots the function I^*, where $I^*(U) := \ln\left(\delta'\left(T_1(U)\right)\right)$ and U is a random variable on (S, Σ, P) with a uniform distribution on $(0, 1)$. Agent 1 receives "large" gains with small probability (*gambling*)

Example, continued

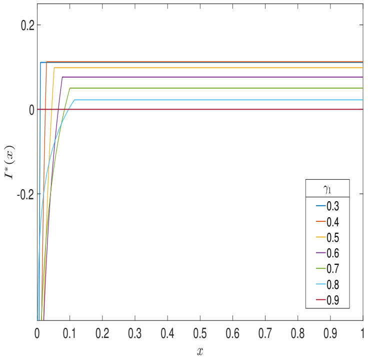


Figure: This graph plots the functions I^*, where $I^*(U) := \ln\left(\delta'\left(T_1(U)\right)\right)$ and U is a random variable on (S, Σ, P) with a uniform distribution on $(0, 1)$. Here, we fix $\gamma_2 = 0.9$, and we vary the parameter γ_1.

Example, continued

$$CEQ_1 := u_1^{-1}\left(\int u_1\left(X_1^*\right) dT_1 \circ P\right) \quad \text{and} \quad CEQ_2 := u_2^{-1}\left(\int u_2\left(X_2^*\right) dT_2 \circ P\right),$$

with $W = 0$.

γ_1	0.3	0.4	0.5	0.6	0.7	0.8	0.9
CEQ_1	7.66%	5.69%	3.53%	1.83%	0.72%	0.14%	0.00%
CEQ_2	7.63%	5.72%	3.24%	1.81%	0.71%	0.14%	0.00%

Table: The certainty equivalents CEQ_1 and CEQ_2 of I^*, where $I^*(U) := \ln\left(\delta'\left(T_1(U)\right)\right)$ and U is a random variable on (S, Σ, P) with a uniform distribution on $(0, 1)$. Here, we fix $\gamma_2 = 0.9$, and we vary the parameter γ_1.

Example, Prelec-1

Prelec-1 (1998) probability weighting function:

$$T_i(t) = \exp(-(-\ln(t))^{\alpha_i}), \;\; \forall t \in [0, 1],$$

for some $\alpha_i > 0$.

- inverse-S shaped when $\alpha_i \in (0, 1)$ and S-shaped when $\alpha_i \geqslant 1$.
- $\Psi(t) = 1 - \exp(-(-\ln(1 - \exp(-(-\ln(t))^{1/\alpha_1})))^{\alpha_2})$.

Example, Prelec-1

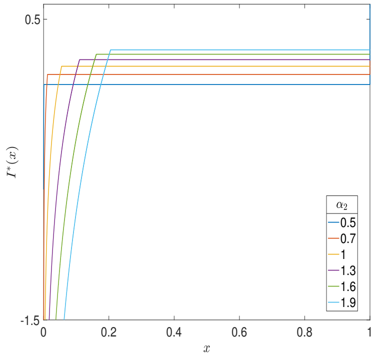


Figure: This graph plots the functions I^*, where $I^*(U) := \ln\left(\delta'\left(T_1(U)\right)\right)$ and U is a random variable on (S, Σ, P) with a uniform distribution on $(0, 1)$. Here, $\alpha_1 = 0.4$, and we vary the parameter α_2.

Example, Prelec-1

α_2	0.7	1	1.3	1.6	1.9
CEQ_1	11.74%	15.47%	19.06%	22.23%	25.00%
CEQ_2	6.28%	13.17%	18.98%	22.43%	24.57%

Table: The certainty equivalents CEQ_1 and CEQ_2 of I^*, where $I^*(U) := \ln\left(\delta'\left(T_1(U)\right)\right)$ and U is a random variable on (S, Σ, P) with a uniform distribution on $(0, 1)$. Here, we fix $\alpha_1 = 0.4$, and we vary the parameter α_1.

Sunspots

Theorem

The following are equivalent:

(1) $\Psi(t) := \widetilde{T}_2\left(T_1^{-1}(t)\right) \geqslant t$ *for all* $t \in [0,1]$.

(2) *There exists a Pareto optimal no-betting allocation.*

(3) *Any Pareto optimal allocation is a no-betting allocation.*

(4) *Every no-betting allocation is Pareto optimal.*

Here, $\Psi(t) := \widetilde{T}_2\left(T_1^{-1}(t)\right) \geqslant t$ writes as

$$T_1(z) + T_2(1-z) \leqslant 1, \text{ or } T_1(z) - z + T_2(1-z) - (1-z) \leqslant 0, \text{ for all } z \in [0,1].$$

For instance, if for a small $z \in (0,1)$, Agent 1 over-weights good outcomes ($T_1(z) > z$) and Agent 2 under-weights bad outcomes ($1 - T_2(1-z) < z$), there is a desire to shift losses from Agent 1 to Agent 2, and thus random Pareto allocations appear.

Sunspots

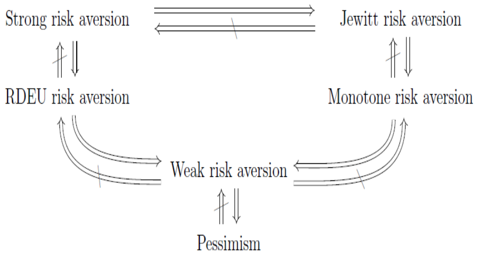
Corollary

If both T_1 and T_2 are convex, then $\Psi(t) \geqslant t$ for all $t \in [0, 1]$.

- Thus, $\Psi(t) \geqslant t$ for all $t \in [0, 1]$ holds when both T_1 and T_2 are linear, and thus when both agents are EU maximizers.

Pessimism

- The *pessimism premium* of Z is given by $\Delta_T(Z) := \int ZdP - \int ZdT \circ P$.
- V is *pessimistic* if $\Delta_T(Z) \geqslant 0$, for all $Z \in L^1(S, \Sigma, P)$.
- **Proposition:** V is pessimistic if and only if $T(t) \leqslant t$, for all $t \in [0, 1]$.
- **Proposition:** If both agents are pessimistic, then $\Psi(t) \geqslant t$ for all $t \in [0, 1]$.



Example, continued

For Tversky and Kahneman (1992)' probability weighting functions:

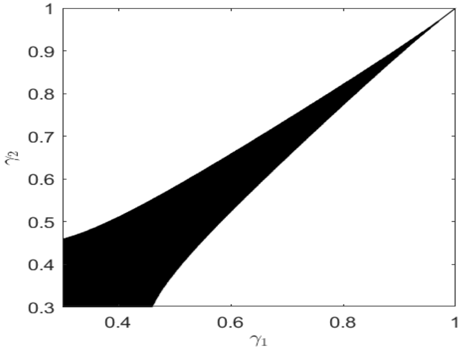
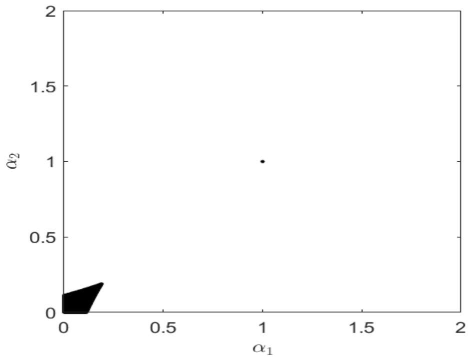


Figure: Only close to the diagonal, there is no betting.

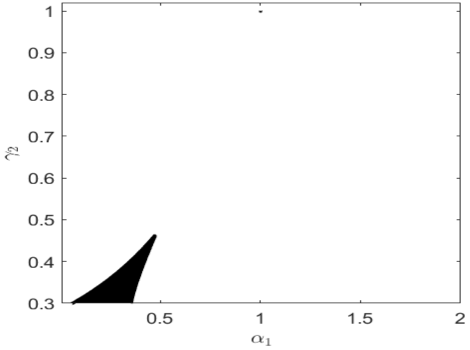
Example, continued

For Prelec-1 (1998)' probability weighting functions:



Example, continued

If Agent 1 is endowed with a Prelec-1 (1998) function, and Agent 2 with a Tversky and Kahneman (1992) function:



Conclusion

We give an explicit characterization of Pareto-optimal allocations, in various situations. In particular, we show that:

(i) Betting is not PO when the two agents are averse to mean-preserving increases in risk (i.e., probability weighting functions are convex).

(ii) If the probability weighting functions are non-convex, then no-betting allocations are PO if it does hold that $\Psi(s) \geqslant s$.

  $\Longrightarrow$ Betting or no betting, this thus *only* follows from probability weighting functions T_i; *not* on the utilities.

(iii) The set of PO is fully described.