

An Integrated Approach to Importance Sampling and Machine Learning for Efficient Monte Carlo Estimation of Distortion Risk Measures in Black Box Models

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Abstract

Distortion risk measures play a critical role in quantifying risks associated with uncertain outcomes. Accurately estimating these risk measures in the context of computationally expensive simulation models that lack analytical tractability is fundamental to effective risk management and decision making. In this paper, we propose an efficient importance sampling method for distortion risk measures in such models that reduces the computational cost through machine learning. We demonstrate the applicability and efficiency of the Monte Carlo method in numerical experiments on various distortion risk measures and models.

Keywords: distortion risk measures; importance sampling; quantile estimation; asset-liability management; monetary risk measures

1 Introduction

Many real world simulation models are typically highly complex. Controlled random inputs are transformed by functions that require costly evaluations. These models also provide the basis for risk measurement and control of firms and systems, and this requires a careful analysis of rare events. Important industry examples are internal models of banks and insurance companies that are applied in their internal risk management process and solvency regulation.

The goal of this paper is to develop an importance sampling algorithm for computing an important class of measures of the downside risk, distortion risk measures (DRMs), when very costly computations are required in the mapping of model inputs to outputs. We call such models *black models* which are characterized by high computational complexity – sometimes even by opaque mechanisms obscuring the relationship between input and output variables. Besides machine learning techniques, our simulation approach builds on two main ingredients:

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(i) efficient importance sampling for quantiles which was developed by Glynn (1996) and Ahn & Shyamalkumar (2011), and (ii) representations of distortion risk measures a mixtures of quantiles, cf. Dhaene et al. (2012).

The quantitative characterization of the downside risk has been studied systematically since the 1990s; the axiomatic approach goes back to Artzner et al. (1999), Föllmer & Schied (2002), and Frittelli & Gianin (2002). An important and extensive class of risk measures are distortion risk measures (DRMs). Particular examples are many distribution-based coherent risk measures, but also commonly used non-convex risk measures. These include value at risk, average value at risk, also called expected shortfall, and range value at risk. Also the insurance premium principles of Wang fall into this class, cf. Wang (1995), Wang (1996).

The main innovations of this paper are:

- (i) Using machine learning techniques, we design an importance sampling algorithm for the Monte Carlo estimation of DRMs in black box models. The construction of an importance sampling distribution requires a discretization of DRM mixtures of quantiles, suitable measure changes for quantiles at different levels, and an efficient allocation of the available samples to these levels. Machine learning provides a cheaper alternative for the highly costly evaluation of the black box model.
- (ii) We analyze and illustrate the performance of the method in various case studies. For DRMs that focus on extremely rare scenarios, we additionally suggest and test an iterative refinement. Finally, the algorithm is successfully applied in a simple asset-liability management model of an insurance firm.

Literature

An extensive treatment of risk measures and DRMs can be found in Föllmer & Schied (2016) and Föllmer & Weber (2015). DRMs are closely related to the concept of Choquet integrals introduced in Choquet (1954) and discussed in detail in Denneberg (1994). DRMs are, for example, studied in Wang (1995), Wang (1996), Kusuoka (2001), Acerbi (2002), Dhaene et al. (2006), Song & Yan (2006), Song & Yan (2009a), Song & Yan (2009b), Weber (2018) and Kim & Weber (2022). The representation theorem for DRMs used in the paper can be found in Dhaene et al. (2012).

Surveys on Monte Carlo simulation and importance sampling are Glasserman (2003) and

Asmussen & Glynn (2007). Importance sampling techniques for rare event simulation are, e.g., discussed in Rubino & Tuffin (2009), Bucklew (2004), Blanchet & Glynn (2008), Dupuis & Wang (2002), Hult & Nyquist (2016), Asmussen, Binswanger & Højgaard (2000), and Juneja & Shahabuddin (2006). More closely related to DRMs are the following papers. The asymptotic properties of the importance sampling quantile estimators we use in this paper are discussed in Glynn (1996) and generalized in Ahn & Shyamalkumar (2011). Glynn (1996) considers the IS estimation of quantiles and suggests four estimators for which asymptotic normality is shown. Results from the theory of large deviations motivate in applications the choice of IS distributions within an exponential class. Ahn & Shyamalkumar (2011) build on this contribution and study IS for $V@R$ and $AV@R$. Asymptotic normality is proven under weaker conditions. Arief et al. (2021) considers rare-event simulation in black box systems focusing on the estimation of probabilities. Glasserman, Heidelberger & Shahabuddin (2002) study the IS estimation of $V@R$ for heavy-tailed risk factors on the basis of exponential measure changes. Dunkel & Weber (2007) investigate the estimation of utility-based shortfall risk, combining stochastic approximation and IS. Brazauskas et al. (2008) focus on the estimation of conditional value at risk, but do not consider IS. The paper proves, for example, the consistency of the estimator and constructs confidence intervals. Sun & Hong (2009) study IS for value at risk and average value risk, exploiting the OCE representation of average value at risk which is due to Rockafellar & Uryasev (2000), Rockafellar & Uryasev (2002), see also Ben-Tal & Teboulle (2007). Measure changes are selected from an exponential family. Beutner & Zähle (2010) present a modified functional delta method for the estimation of DRMs and derive asymptotic distributions and approximate confidence intervals, mainly motivated from a statistical perspective. They do not consider IS. Pandey, Prashanth & Bhat (2021) combine a trapezoidal rule and quantile estimation to estimate spectral risk measures. Bounds for the error in probability are proven. IS is not considered. Estimators of DRMs are often related to L -estimators for which we refer to Stigler (1974) and Serfling (1980).

Surveys on techniques and applications of machine learning are Shalev-Shwartz & Ben-David (2014) and Mohri, Rostamizadeh & Talwalkar (2018). Some applications of black box models in finance are discussed in Huang, Chai & Cho (2020).

Outline

The paper is organized as follows: In Section 2 we set the scene by briefly introducing DRMs, the considered quantile estimators, and their asymptotic distribution. The importance sampling method for DRMs is also developed in this section. Section 3 applies the method in various case studies to test its performance. Section 4 discusses an application to asset-liability management of an insurance firm. Auxiliary results are collected in an online appendix. This includes background material on distortion risk measures, asymptotics of quantile estimators in importance sampling, tools from machine learning, some computations and proofs, and additional figures on the basis of data that were obtained in the case studies.

2 Efficient Estimation of DRM of Black Box Models

2.1 Setting the Scene

Measuring risk in complex systems is an important task. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \rightarrow \mathbb{R}^d$ a random vector. The random outcome of the system is modeled by a random variable $Y = h(X)$ for some measurable function $h : \mathbb{R}^d \rightarrow \mathbb{R}$. We assume that Y is accessible via a simulation oracle, but that h highly complex and not analytically tractable. In contrast, the distribution of the random vector X is explicitly known and can appropriately be modified in order to increase the efficiency of the estimation. The assumption is that even if the simulation mechanism for X is changed, the function h can still be evaluated, but its evaluation is very costly. The problem is to determine $\rho(h(X))$ by simulation where ρ is a monetary risk measure. Our sign convention is that $h(X)$ counts losses as positive and gains as negative, as is customary in actuarial science. More specifically, we suppose that $\rho = \rho_g$ is a distortion risk measure (DRM) associated to a distortion function $g : [0, 1] \rightarrow [0, 1]$ of the form $\rho_g(Y) = \int_{-\infty}^0 [g(\mathbb{P}(Y > y)) - 1] dy + \int_0^{\infty} g(\mathbb{P}(Y > y)) dy$. For further details, we refer to Appendix A.1. By Dhaene et al. (2012), see also Bettels, Kim & Weber (2022), DRMs can be written as mixtures of quantiles, i.e.,

$$\rho_g(Y) = c_1 \int_{[0,1]} q_Y^+(1-u) dg_1(u) + c_2 \int_{[0,1]} q_Y(1-u) dg_2(u), \quad (1)$$

where g_1, g_2 are right- resp. left-continuous distortion functions, $c_1 + c_2 = 1$, $c_1, c_2 \in [0, 1]$, $g = c_1 g_1 + c_2 g_2$, and $q_Y^+(u) = \sup\{y | F_Y(y) \leq u\}$, $q_Y(u) = \inf\{y | F_Y(y) \geq u\}$. Eq. (1) is the

starting point for the Monte Carlo simulation scheme.

The risk estimation problem has two aspects: The quantiles, which appear as integrands in eq. (1), must be simulated efficiently, and the integrals must be discretized. We propose an importance sampling technique for the quantile estimation procedure based on machine learning estimation (ML) of the function h ; in addition, we devise a strategy for allocating samples along the discretization to achieve good performance. The next sections explain how to design and to implement the following algorithm for the Monte Carlo estimation of DRMs.

Algorithm 1 Importance Sampling DRM Estimation Algorithm

1: **Input:** Distortion function g , pivot sample size M , sample size N , size of partition m .

2: **Output:** Estimation of $\rho_g(Y)$

3: **function** MAIN:

4: Set $\alpha_i = i\alpha/m$ for $i \in \{0, \dots, m\}$ and $\alpha_{m+1} = 1$;

5: Sample $X \leftarrow (X_1, \dots, X_M)$ from F and set $Y \leftarrow (h(X_1), \dots, h(X_M))$;

6: **for** $i \in \{0, \dots, m\}$ **do**

7: Set $aux \leftarrow$ empirical quantile of sample Y at level $1 - \alpha_i$;

8: Set ϑ_i such that

$$aux = \frac{\sum_{j=1}^M Y_j \exp(\vartheta_i Y_j)}{\sum_{j=1}^M \exp(\vartheta_i Y_j)};$$

9: **for** $i \in \{0, \dots, m\}$ **do**

10: Set $aux \leftarrow$ empirical quantile of sample Y at level $1 - \alpha_i$;

11: Set $aux_c \leftarrow \frac{1}{M} \sum_{j=1}^M \frac{dF}{dF_{\vartheta_i}}(X_j) \mathbb{1}_{\{Y_j > aux\}}$;

12: Set c_i such that

$$c_i \leftarrow \frac{aux_c - \alpha_i^2}{G'_Y(aux)} \cdot (g(\alpha_{i+1}) - g(\alpha_i));$$

13: **for** $i \in \{0, \dots, m\}$ **do**

14: Set

$$p_i \leftarrow \frac{\sqrt{c_i}}{\sum_{i=0}^m \sqrt{c_i}};$$

15: Choose \hat{h} as the regression selected by a k -fold validation and calibration from X, Y ;

16: Set F_i for $i \in \{0, 1, \dots, m\}$ such that

$$dF_i = \exp\left(\vartheta_i \hat{h}(x) - \hat{\psi}(\vartheta_i)\right) dF;$$

17: Sample $\theta_1, \dots, \theta_N$ as i.i.d. copies of θ such that $P(\theta = i) = p_i$ for $i \in \{0, 1, \dots, m\}$;

18: Sample $X' \leftarrow (X'_1, \dots, X'_N)$ such that $X_i \sim F_{\theta_i}$ and set $Y' \leftarrow (h(X'_1), \dots, h(X'_N))$;

19: Set $estimate \leftarrow 0$;

20: **for** $i \in \{0, \dots, m\}$ **do**

21: **Option 1:** Compare the variances of $\hat{q}_{F_i, N_i}(1 - \alpha_i)$ and $\hat{q}_{F^*, N}(1 - \alpha_i)$;

22: Set $\hat{q}_Y(1 - \alpha_i)$ as the better performing estimator;

23: **Option 2:** Set $\hat{q}_Y(1 - \alpha_i) \leftarrow \hat{q}_{F^*, N}(1 - \alpha_i)$;

24: Set $estimate \leftarrow estimate + \hat{q}_Y(1 - \alpha_i) \cdot (g(\alpha_{i+1}) - g(\alpha_i))$;

25: **Return:** $estimate$;

2.2 Quantile Estimation with Importance Sampling

We begin with a quantile estimation technique that incorporates importance sampling, as proposed and studied in Glynn (1996) and Ahn & Shyamalkumar (2011). For this we will first assume that the function h is known; ML techniques in the simulation are discussed in Section 2.4.

Let F be the distribution function of X under \mathbb{P} , and F^* some other distribution function on \mathbb{R}^d such that F is absolutely continuous with respect to F^* with Radon-Nikodym density $\frac{dF}{dF^*}$. Sampling $(X_i)_{i=1,2,\dots,N}$ independently from F^* , for any $u \in (0, 1)$ a quantile estimator of $q_Y(u)$ is given by

$$\hat{q}_{F^*,N}(u) := \inf \left\{ x \in \mathbb{R} \left| \frac{1}{N} \sum_{h(X_i) > x} \frac{dF}{dF^*}(X_i) \leq 1 - u \right. \right\}, \quad u \in (0, 1). \quad (2)$$

Conditions for the asymptotic normality of this estimator are provided in Glynn (1996) and Ahn & Shyamalkumar (2011) and stated in Appendix A.4. More precisely, denoting by $G_Y = F \circ h^{-1}$ and $G^* = F^* \circ h^{-1}$ the distribution functions of $h(X)$ under F and F^* , respectively, Ahn & Shyamalkumar (2011) show the following result.

Theorem 2.1. *Suppose that Assumption A.12 holds. Then for $u \in (0, 1)$ we have*

$$\sqrt{N} (\hat{q}_{F^*,N}(u) - q_Y(u)) \xrightarrow[N \rightarrow \infty]{d} \mathcal{N} \left(0, \frac{\mathbb{E}_{F^*} \left[\frac{dF}{dF^*}(X)^2 \mathbb{1}_{\{h(X) \in (q_Y(u), \infty)\}} \right] - (1 - u)^2}{G'(q_Y(u))^2} \right).$$

This theorem can now be leveraged to construct a sampling distribution F^* that improves the efficiency of the Monte Carlo simulation. A classical choice for large, rare outcomes are exponential tilts. In our setting, X is sampled, and we are interested in large outcomes of $Y = h(X)$. We thus consider the candidate family of sampling distribution

$$dF_\vartheta = \exp(\vartheta h(x) - \psi(\vartheta)) dF(x), \quad (3)$$

where $\vartheta \in \Theta \subseteq \mathbb{R}$, $\psi(\vartheta) := \log(\mathbb{E}_F[\exp(\vartheta h(X))])$. In order to minimize the variance in Theorem 2.1, Sun & Hong (2009) minimize a suitable upper bound and obtain the condition

$$q_Y(u) = \mathbb{E}_{F_\vartheta}[h(X)], \quad (4)$$

which is used to determine a good parameter ϑ . They prove that, under suitable technical conditions, the corresponding measure change reduces the variance of the estimator. We review this in Appendix A.4. The implementation of eq. (4) requires knowledge of the quantile $q_Y(u)$ which is the value we seek to estimate; moreover, the exact structure of h and $\psi(\vartheta)$ are unknown. An algorithmic approach based on ML and MCMC to overcome these challenges is presented in Section 2.4.

2.3 Discretization and Optimal Allocation of the Sampling Budget

The estimation of eq. (1) requires a discretization of the two integrals involving the left- and right-continuous distortion functions. This distinction is relevant, if $q_Y^+(1-u) \neq q_Y(1-u)$ for some u . In the numerical implementation, we suppose that $q_Y^+(1-u) = q_Y(1-u)$ for all u that appear in the discretization. This condition is consistent with the application of Theorem 2.1 and Assumption A.12 that forms the basis for the quantile approximation that we use. The requirement is essentially that the distribution function grows locally in a neighborhood of the quantiles.

We consider the approximation

$$\hat{\rho}_g(Y) = \sum_{i=0}^m \hat{q}_{F_i, N_i}(1 - \alpha_i)(g(\alpha_{i+1}) - g(\alpha_i)) \quad (5)$$

of $\rho_g(Y)$, defined for a partition $0 = \alpha_0 < \alpha_1 < \dots < \alpha_m < \alpha_{m+1} = 1$ where $\hat{q}_{F_i, N_i}(1 - \alpha_i)$ are importance sampling estimators according to Section 2.2 with the sampling distributions F_i and an allocated sampling budget N_i . For example, the partition $(\alpha_i)_{i=0,1,\dots,m}$ could be chosen uniform in the region where g grows, or one could choose $\alpha_i = g^{-1}\left(\frac{i}{m+1}\right)$, $i = 0, 1, \dots, m+1$, to adequately cover the levels where g places weight. We suppose that the technical Assumption A.12 is satisfied and that for each i the number of samples N_i is chosen large enough so that the $\hat{q}_{F_i, N_i}(1 - \alpha_i)$ is finite according eq. (7). Using the notation

$$\hat{q}_Y(1-u) := \sum_{i=0}^m \mathbb{1}_{\{u \in [\alpha_i, \alpha_{i+1})\}} \hat{q}_{F_i, N_i}(1 - \alpha_i)$$

for the estimator of the quantile function, we obtain $\hat{\rho}_g(Y) = \int_0^1 \hat{q}_Y(1-u) dg(u)$.

Two questions have to be considered: First, how should the available N samples be allocated to each quantile at the different levels? Second, should individual importance sampling be used

for each quantile, or should a single common measure change with pooled samples be preferred?

2.3.1 Sample Allocation to Quantiles

Using Jensens's inequality, Fubini's theorem and Theorem 2.1 the MSE of the estimator can approximately be bounded above as follows:

$$\begin{aligned} \mathbb{E} [(\rho_g(Y) - \hat{\rho}_g(Y))^2] &= \mathbb{E} \left[\left(\int_0^1 q_Y(1-u) - \hat{q}_Y(1-u) dg(u) \right)^2 \right] \leq \left(\int_0^1 \mathbb{E}[(q_Y(1-u) - \hat{q}_Y(1-u))^2] dg(u) \right) \\ &\approx \sum_{i=0}^m \int_{\alpha_i}^{\alpha_{i+1}} (q_Y(1-u) - q_Y(1-\alpha_i))^2 dg(u) + \underbrace{\frac{\mathbb{E}_{F_i} \left[\frac{dF}{dF_i}(X)^2 \mathbb{1}_{\{h(X) > q_Y(1-\alpha_i)\}} \right] - \alpha_i^2}{N_i G'_Y(q_Y(1-\alpha_i))^2}}_{:=S_i(N_i)} (g(\alpha_{i+1}) - g(\alpha_i)). \end{aligned}$$

In the latter sum only the second summand $S_i(N_i)$ depends on N_i . Hence, the optimal allocation is obtained by minimizing $\sum_{i=0}^m S_i(N_i)$ under the constraint $\sum_{i=0}^m N_i = N$. This leads (up to rounding) to the solution

$$N_i^* = N \frac{\sqrt{c_i}}{\sum_{j=0}^m \sqrt{c_j}}, \quad i = 1, 2, \dots, m, \quad (6)$$

where $c_j := \frac{\mathbb{E}_{F_j} \left[\frac{dF_j}{dF_j}(X)^2 \mathbb{1}_{\{h(X) > q_Y(1-\alpha_j)\}} \right] - \alpha_j^2}{G'_Y(q_Y(1-\alpha_j))} (g(\alpha_{j+1}) - g(\alpha_j))$, $j \in \{0, 1, \dots, m\}$. The derivation of this result can be found in Appendix A.6.1.

If the total sample size N is not known in advance, eq. (6) determines the fraction $p_i := \frac{N_i^*}{N}$ of the samples generated for each quantile. The total collection of all samples for these quantiles can also be viewed as samples from the mixture sampling distribution $F^* := \sum_{i=0}^m p_i \cdot F_{\vartheta_i^*}$, where $F_{\vartheta_i^*}$ are the sampling distributions for each individual quantile constructed in Section 2.2.

2.3.2 Efficient Use of the Samples in the Estimation of Multiple Quantiles

The estimation of DRMs according to eq. (5) requires the estimation of quantiles at the levels $1-u$ for $u = \alpha_0, \dots, \alpha_m$. We discuss whether individual importance sampling should be used or a single common measure change with pooled samples is preferred. We assume in our comparison that the generation of individual samples is costly, but that the evaluation of the quantile estimators for given samples is comparatively inexpensive. We suppose that samples are allocated to the individual quantiles according to eq. (6) and that F^* is the mixture sampling distribution in Section 2.3. For each i , estimators of $q_Y(1-\alpha_i)$ are $\hat{q}_{F_{\vartheta_i^*}, N_i^*}(1-\alpha_i)$

and $\hat{q}_{F^*,N}(1 - \alpha_i)$ with approximate variances

$$\frac{V(1 - \alpha_i, F_{\vartheta_i^*})}{N_i^*} = \frac{\mathbb{E}_{F_{\vartheta_i^*}} \left[\frac{dF}{dF_{\vartheta_i^*}}(X)^2 \mathbb{1}_{\{h(X) > q_Y(1 - \alpha_i)\}} \right] - \alpha_i^2}{N_i^* G'_Y(q_Y(1 - \alpha_i))^2},$$

$$\frac{V(1 - \alpha_i, F^*)}{N} = \frac{\mathbb{E}_{F^*} \left[\frac{dF}{dF^*}(X)^2 \mathbb{1}_{\{h(X) > q_Y(1 - \alpha_i)\}} \right] - \alpha_i^2}{N G'_Y(q_Y(1 - \alpha_i))^2},$$

respectively, if the conditions of Theorem 2.1 are satisfied. Hence, by comparing

$$\frac{1}{N_i^*} \mathbb{E}_{F_{\vartheta_i^*}} \left[\frac{dF}{dF_{\vartheta_i^*}}(X)^2 \mathbb{1}_{\{h(X) > q_Y(1 - \alpha_i)\}} \right] \text{ to } \frac{1}{N} \mathbb{E}_{F^*} \left[\frac{dF}{dF^*}(X)^2 \mathbb{1}_{\{h(X) > q_Y(1 - \alpha_i)\}} \right],$$

the preferred estimator can be selected. For m not too small, the variance corresponding to the mixture distribution will typically be smaller, since $N_i^*/N = \mathcal{O}(m)$ for all i . For this reason, we propose using the mixture distribution and implement it in all case studies in Sections 3 & 4.

2.4 Implementation Issues

The object of interest is $\rho_g(Y)$ for $Y = h(X)$, where we assume that the evaluation of the function h is highly costly. In order to apply importance sampling to this situation, we propose to, first, compute the parameters governing the exponential changes of measure by using pivot samples. Second, h is approximated by some auxiliary function \hat{h} simplifying the changes of measure and accelerating the simulation of the corresponding samples. These are typically acceptance-rejection methods, where we avoid the costly evaluation of $Y = h(X)$ when selecting accepted samples. Third, quantile estimators are computed for these samples for the original random variable $Y = h(X)$.

Letting $(X_1, h(X_1)), \dots, (X_M, h(X_M))$ denote pivot samples, we estimate ϑ_i^* by $\hat{\vartheta}_i^*$ according to eq. (4), i.e., solving $\hat{q}_{F,M}(1 - \alpha_i) = \frac{\sum_{i=1}^M h(X_i) \exp(\hat{\vartheta}_i^* h(X_i))}{\sum_{i=1}^M \exp(\hat{\vartheta}_i^* h(X_i))}$ numerically where $\hat{q}_{F,M}(1 - \alpha_i)$ is the crude quantile estimator. Estimating $\psi(\vartheta_i^*)$ from the empirical measure gives $\hat{\psi}_i = \log \left(\frac{1}{M} \sum_{j=1}^M \exp(\hat{\vartheta}_i^* h(X_j)) \right)$. In order to facilitate the generation of samples, we approximate h by \hat{h} using ML-techniques on the pivot samples. More specifically, we consider linear and polynomial predictors, linear, polynomial and Gaussian support vector machines (SVMs) and k -nearest neighbor (NN) regressions, using k -fold validation to compare methods and parameters to determine an approximation with smallest MSE on splits of the training sample; for a brief review of the methods see Appendix A.5 and Shalev-Shwartz & Ben-David (2014).

Importance sampling is now based on the approximation \hat{h} , i.e., instead of $F_{\vartheta_i^*}$ we use a measure change with a Radon-Nikodym density with respect to f that is proportional to $\exp(\hat{\vartheta}_i^* \hat{h} - \hat{\psi}_i)$. Since $\hat{\psi}_i$ was estimated from h instead of \hat{h} , the resulting density $\hat{f}_{\hat{\vartheta}_i^*}$ may not be correctly normalized. The normalization constant can either be estimated at this stage, or Markov Chain Monte Carlo can be used for the simulation. The latter approach does, of course, typically not preserve the independence of simulations, but, as we will see in our case studies, is quite efficient. Using this approach, the normalizing constants can afterwards be estimated on the basis of additional samples. We can sample from $\hat{f}_{\hat{\vartheta}_i^*}$ or directly from the mixture distribution $p_0 \hat{f}_{\hat{\vartheta}_0^*} + \dots + p_m \hat{f}_{\hat{\vartheta}_m^*}$ with $p_i = N_i^*/N$, applying the Metropolis-Hastings algorithm with random walk proposal.

For the importance sampling quantile estimator (2) we need to evaluate the likelihood ratio which requires knowledge of the normalizing constants. To be more precise, we have $\frac{dF}{d\hat{F}_{\hat{\vartheta}_i^*}} = \frac{c_i}{\exp(\hat{\vartheta}_i^* \hat{h}(x) - \hat{\psi}_i)}$, $c_i = \int \exp(\hat{\vartheta}_i^* \hat{h}(x) - \hat{\psi}_i) f(x) dx$ and $\frac{dF}{d\left(\sum_{i=0}^m p_i \hat{F}_{\hat{\vartheta}_i^*}\right)} = \frac{c}{\sum_{i=0}^m p_i \exp(\hat{\vartheta}_i^* \hat{h}(x) - \hat{\psi}_i)}$, $c = \sum_{i=0}^m p_i c_i$. The normalizing factors c_i and c can be approximated by either using a trapezoidal rule with the $N+M$ samples as grid points or by applying an adaptive quadrature rule explained in Shampine (2008). In the implementations we chose for each application the method which performed best in test cases.

Another approach for estimating a normalizing constant relies on an estimation of the density function from the samples drawn. Consider, for example, the mixture distribution, and let \hat{f} be the estimated density, e.g., via kernel density estimation. Then for all $x \in \mathbb{R}^d$ we have that $c \approx \frac{\sum_{i=0}^m p_i \exp(\hat{\vartheta}_i^* \hat{h}(x)) f_X(x)}{\hat{f}(x)}$. Thus, c can be estimated by computing the right hand side for several x and taking an average.

3 Case Studies

In this section, we apply the developed method to various test models and distributions. The goal is to experimentally evaluate the variance reduction achieved by the proposed algorithm compared to importance sampling in the exact model, which is known in closed form for the test cases. We compare the root mean square errors (RMSEs) when estimating different DRMs induced by concave and convex distortion functions.

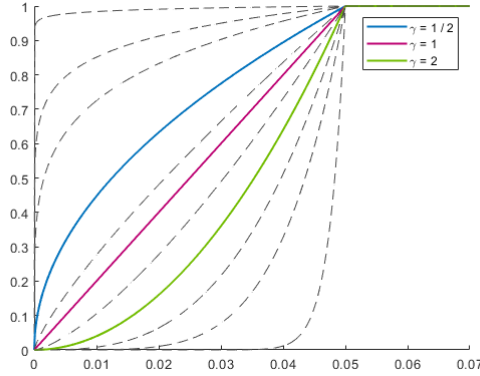


Figure 1: Different distortion functions $g_{\alpha, \gamma}(u)$ with $\alpha = 0.05$ and $\gamma \in \{0.01, 0.1, 0.2, 0.5, 0.8, 1, 1.4, 2, 3, 5, 20\}$. The distortion function used in the case studies are highlighted.

3.1 Simulation Design

We consider the distortion function $g_{\alpha, \gamma}(u) = \mathbf{1}_{\{u \in [0, \alpha]\}} \left(\frac{u}{\alpha}\right)^{1/\gamma} + \mathbf{1}_{\{u \in (\alpha, 1]\}}$ with $\gamma \in \{1/2, 1, 2\}$ illustrated in Figure 1, see Example A.6 in the Appendix for more details. The concave function $g_{2, \alpha}$ defines a convex DRM, $g_{1/2, \alpha}$ is convex, and $g_{1, \alpha}$ corresponds to the AV@R at level α .

In our numerical experiments, we repeat Algorithm 1 for DRM estimation R times to obtain data that can be further analyzed. For clear benchmarking, we specify the hypothesis class used in each of these experiments and compare the results across hypothesis classes. Thus, we do not perform the k -fold validation in line 15 of Algorithm 1 in each repetition, but only recalibrate within any previously selected class. Additionally, we identify the winner of a k -fold validation with $k = 20$ based on $M = 2000$ pivot samples. Numerical tests in the context of our case studies show that this determination of a ML hypothesis class is quite robust, i.e., different sets of $M = 2000$ pivot samples typically lead to the selection of the same class.

We consider the following functions and distributions:

- (1) *Identity of Normal*: We set $X \sim \mathcal{N}(0, 1)$ and $h(x) = x$, implying $Y \sim \mathcal{N}(0, 1)$. The k -fold cross validation from the pivot samples suggests using a linear regression to approximate h .
- (2) *Sum of Normals*: We consider $X_1, X_2 \sim \mathcal{N}(0, 1)$ with $\text{Corr}(X_1, X_2) = 0.3$ and $h(x_1, x_2) = x_1 + x_2$. The k -fold cross validation suggests a linear regression.
- (3) *Product of Normals*: Let $X_1, X_2 \sim \mathcal{N}(2, 1)$ with $\text{Corr}(X_1, X_2) = -0.3$ and $h(x_1, x_2) = x_1 \cdot x_2$. The k -fold validation identifies the SVM with a polynomial kernel of degree 2 as

the best choice for \hat{h} .

- (4) *Sum of Squared Normals*: Consider the independent random variables $X_1, X_2, X_3, X_4 \sim \mathcal{N}(0, 1)$ and let $h(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2$. Then $h(X_1, X_2, X_3, X_4)$ follows a χ^2 distribution with 4 degrees of freedom. The k -fold validation suggests a polynomial support vector machine with degree 2.
- (5) *Sine and Uniform*: Set $X \sim \text{unif}(0, 1)$ and $h(x) = x \sin(2.5 \cdot \pi \cdot x)$. This example is used in Altman (1992) to illustrate k -NN regression. The k -fold validation suggests polynomial regression with degree 5 as an approximation of h .
- (6) *Logistic Transformation and Exponential*: Letting $X \sim \exp(1)$, then $h(X)$ with $h(x) = -\log(e^{-x}/(1 - e^{-x}))$ follows a Logistic(0, 1) distribution. The k -fold validation suggests either a support vector machine with Gaussian kernel or k -NN regression with $k = 1$.

For each of these functions and distributions, we perform numerical experiments for different ML hypothesis classes used to approximate h in the importance sampling algorithms. In particular, this analysis is also performed for the winner of the k -fold validations. Each experiment is repeated R times. In all cases, we implement a crude estimation with $M + N$ samples as well as an estimation based on the importance sampling method with M pivot samples and N samples from the mixture distribution defined in Section 2.3. As a benchmark, we determine an “exact value” by a crude estimation with 10,000,000 samples. From the replications we obtain for all cases an estimated root mean-square error (RMSE).

3.2 Results

3.2.1 Distribution of the Samples

To illustrate the measure change, we consider model (5) in Figure 2. As mentioned before, option 2 in line 23 of Algorithm 1 was implemented in all case studies. An analogous analysis for models (1)-(4) & (6) can be found in Figures 6 & 7. We consider the DRMs $\rho_{g_{\alpha, \gamma}}(Y)$, $\alpha = 0.05, \gamma \in \{1/2, 1, 2\}$. The figures show the true density of Y . Additionally, 200 samples from the mixture distribution (with values on the x-axis) are plotted along the probability density. The labeled quantile $q_Y(0.05)$ indicates the threshold above which samples are relevant for the estimation of $\rho_{g_{\alpha, \gamma}}(Y)$. The crosses mark the expectations of the individual importance sampling

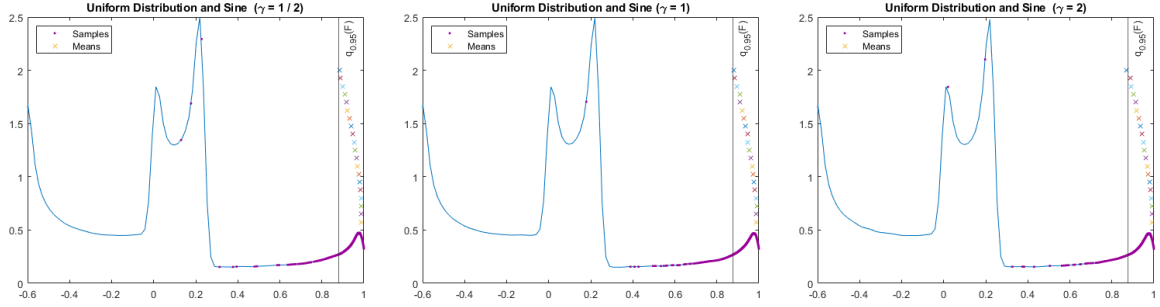


Figure 2: An example of 200 samples drawn from the mixture distribution plotted on the underlying distribution of the model Y for the case study (5). To approximate the mixture weights and optimal mixture components $M = 20,000$ pivot samples were drawn.

components of the mixture distribution. Samples and expectations are, by construction, in the right tail of the distribution in the area relevant for the estimation of the DRM.

3.2.2 Efficiency of the Estimations

In this section, we discuss the efficiency of the algorithm for case studies (1)-(6). The ML approximation used in importance sampling is fixed, and we compare different hypothesis classes. The calibration of the ML regressions and the construction of the importance sampling measure change are based on $M = 2,000$ pivot samples. We choose $m = 20$ and $N = 20,000$ and run $R = 1,000$ replications of the simulation for estimating the RMSE. In Figure 3 & 4, the ratio of the RMSE between the crude estimate and the proposed importance sampling method for the DRMs $\rho_{g_{\alpha,\gamma}}$ with $\gamma \in \{1/2, 1, 2\}$ and $\alpha \in [0.01, 0.3]$ for models (1) to (6). The absolute estimated RMSE for the different estimation methods is shown in Figure 12 & 13 in Appendix A.7.

The plots confirm that the proposed importance sampling algorithm can successfully reduce the RMSE in all cases. The efficiency of the algorithm depends strongly on the ML regression chosen. A poorly chosen approximation can even lead to a higher error than the crude method. Interestingly, the choice based on k -fold validation and pivot calibration works reasonably well in all cases, despite the fact that the ML objective function does not focus on the tail. The smaller α , the more the DRM zooms in on the tail risk due to rare events. As expected, the variance reduction becomes better the smaller α is. Similarly, variance reduction is also better the smaller γ , since DRMs with smaller γ put more emphasis on tail risk.

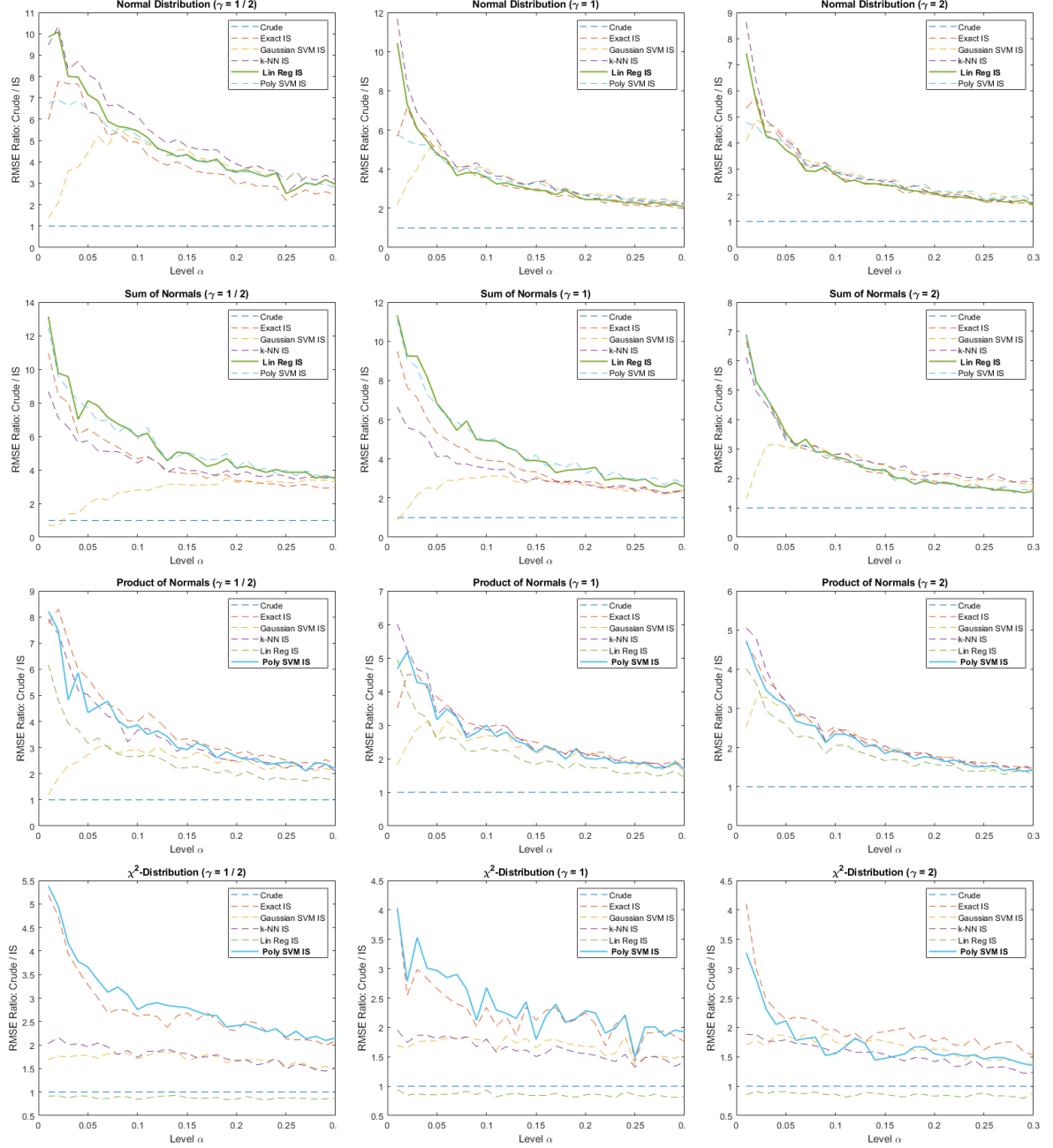


Figure 3: Ratio of the RMSE arising between the crude method and the importance sampling method when estimating $\rho_{g_{\gamma,\alpha}}$, with $\gamma \in \{1/2, 1, 2\}$, $\alpha \in [0.01, 0.3]$, for the models (1) to (6). The comparison is made between the crude method and the proposed importance sampling method with the different approximations for the black box considered in the paper.

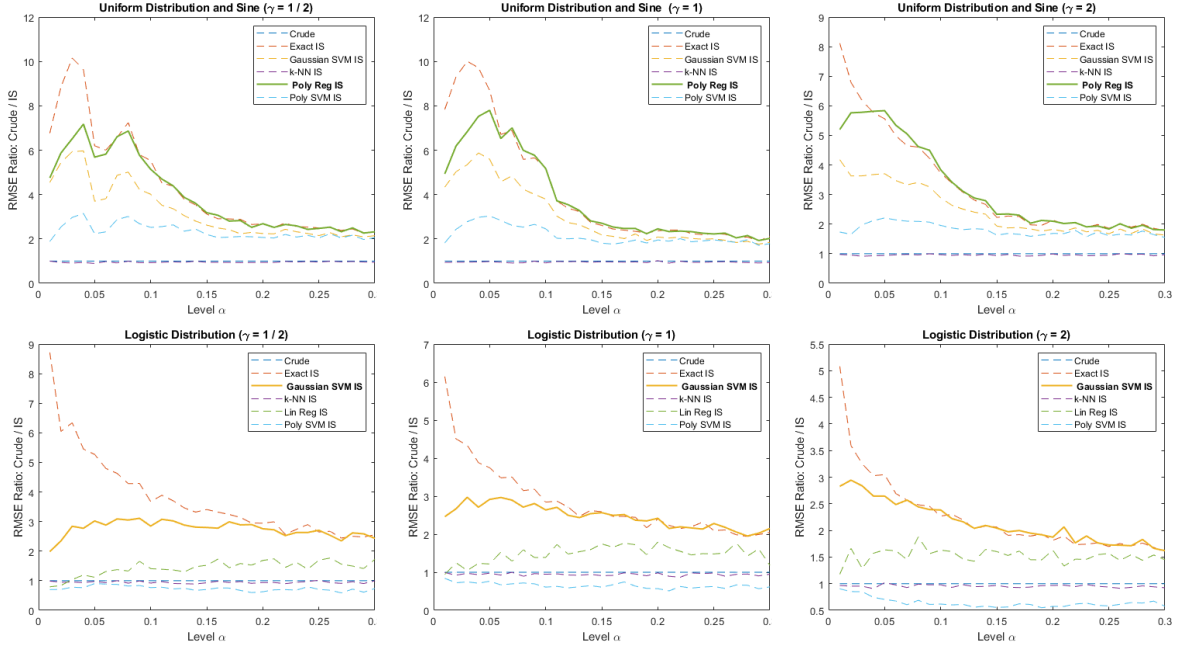


Figure 4: Continuation of Figure 3.

3.3 Iterative Exploration of the Extreme Tail

The discussed algorithm consists of two simulation steps. First, pivot samples are drawn that are used for both the choice of the ML approximation and the determination of an IS measure change. Second, samples are generated under the IS distribution and used for the estimation of the DRMs. However, if DRMs are considered that focus on particularly extreme tail events, this approach might not yet be sufficient. A possible extension to the suggested approach is the following: The samples from the IS distribution are not directly used for DRM estimation, but serve as additional data for calibrating the ML approximation a second time and the construction of a further measure change on this basis. In this section, we provide a case study that takes this approach – the iterative exploration of the extreme tail.

Simulation Design

We consider again the case studies (1) - (4) outlined in Section 3.1 with the same distortion functions $g_{\alpha,\gamma}(u)$, $\gamma \in \{1/2, 1, 2\}$ and corresponding DRMs. We do, however, focus, on more extreme tail events by choosing $\alpha = 0.002$. In addition, we choose a finer partition by setting $m = 50$. In the experiment, we repeat all simulation runs $R = 2,000$ times to estimate the RMSE. We compare three simulation approaches for the estimation of the DRMs. In all cases, 27,500 samples are used, respectively.

	(1) Id. of Normals			(2) Sum of Normals			(3) Prod. of Normals			(4) Sum of Sq. Normals		
γ	1/2	1	2	1/2	1	2	1/2	1	2	1/2	1	2
Exact	3.35	3.16	3.02	5.40	5.09	4.88	4.08	3.60	3.30	20.58	19.01	17.99
Mean CRUDE	3.33	3.15	3.03	5.38	5.08	4.88	4.02	3.59	3.30	20.45	18.99	18.00
Mean IS	3.35	3.16	3.02	5.40	5.09	4.88	4.07	3.60	3.30	20.48	18.91	17.89
Mean ITER IS	3.35	3.16	3.02	5.40	5.09	4.88	4.06	3.59	3.29	20.55	18.98	17.97
RMSE $\frac{\text{CRUDE}}{\text{IS}}$	35.61	20.63	14.87	16.73	13.09	10.34	10.90	9.40	7.48	5.34	3.58	2.97
RMSE $\frac{\text{CRUDE}}{\text{ITER IS}}$	35.42	20.81	14.41	35.02	21.70	15.04	9.61	8.39	6.71	13.55	8.89	7.17

Table 1: Results of the DRM estimation in the extreme tail.

The first approach is a crude simulation with 27,500 samples. The second approach is the algorithm suggested in the previous sections with 7,500 pivot samples and 20,000 samples from the IS distribution. The third approach is an iterative exploration: 5000 pivot samples are used to calculate the IS distribution for level $\alpha' = 0.01$. Then we draw from this IS distribution 2,500 additional pivot samples. The IS distribution for $\alpha = 0.002$ is computed from the total 7,500 pivot samples. In the last step, 20,000 are drawn from this distribution to estimate the DRMs.

Results

The results of the case study are displayed in Table 1. The exact values of the DRMs, the means over $R = 2,000$ simulation runs and the corresponding ratios of the RMSE of the two IS methods and the crude method are documented. Overall the iterative method generally yields the best RMSE reduction, outperforming the direct IS approach substantially in experiments (2) and (4). The direct IS approach is still more efficient than the crude method. Especially in (1) and (2) the reduction of the RMSE is substantial in contrast to the crude method, while in (3) and (4) the IS methods are not as efficient. When considering the mean over all simulation runs, we observe that the IS methods also reduce estimation bias.

4 Application to ALM

We apply Algorithm 1 to the estimation of solvency capital in a simple asset-liability management (ALM) model of an insurance firm. Instead of the risk measure $V@R$ which forms the basis of Solvency II, we use the same DRMs that we considered in Section 3. The suggested method could also be applied in highly complex ALM models such as those applied by major insurance groups.

4.1 Model Description

Our ALM model, inspired by Weber et al. (2014) and Hamm, Knispel & Weber (2019), describes a snapshot in time of an ongoing insurance business. The focus is on a one-year time horizon with dates $t = 0, 1$, as in Solvency II. The values of assets and liabilities are denoted by A_t, L_t , $t = 0, 1$, respectively. At each point in time, their difference is the book value of equity $E_t = A_t - L_t$, $t = 0, 1$, which is used for the solvency capital calculation.

The evolution of balance sheet is driven by market and insurance risks. For simplicity, we assume that reserves are constant, i.e., $L_t = v \forall t$. Any changes in value are thus seen on the asset side. We assume that insurance claims are modeled by a collective model where the number of claims are given by the counting process N and their severities by independent, identically distributed losses ξ_k . Annual total premium payments π are received at the beginning of the year. We set

$$C = \sum_{k=1}^N \xi_k.$$

The random annual return of assets between dates $t = 0$ and $t = 1$ is denoted by R_A , i.e., we obtain that

$$A_1 = R_A \cdot A_0 - C + \pi.$$

In order to model the random return of assets we assume that

$$A_0 = \eta^S S_0 + \eta^B B_0,$$

where $B = (B_t)_{t \in \{0,1\}}$ and $S = (S_t)_{t \in \{0,1\}}$ are the prices of a bond and a stock and η^S and η^B the respective holdings. This implies that

$$R_A = \frac{\eta^S S_1 + \eta^B B_1}{A_0} = \frac{\eta^S S_0}{A_0} \cdot \frac{S_1}{S_0} + \frac{\eta^B B_0}{A_0} \cdot \frac{B_1}{B_0} = b \cdot \frac{S_1}{S_0} + (1-b) \cdot \frac{B_1}{B_0},$$

where b is the fraction of initial wealth invested in the stock. Setting $B_0 = 1$, $B_1 = 1 + R_B$ for some random interest rate $R_B > -1$, and $S_0 = 1$,

$$S_1 = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) \Delta t + \sigma \sqrt{\Delta t} Z\right),$$

we derive that

$$R_A = b \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}Z\right) + (1-b) \cdot (1 + R_B).$$

Solvency capital is determined in terms of risk measure applied to the difference of the net asset values over the time period of one year, i.e.,

$$E_1 - E_0 = A_1 - \underbrace{L_1}_{=v} - A_0 + \underbrace{L_1}_{=v} = (R_A - 1) \cdot A_0 - C + \pi.$$

4.2 Simulation Overview

As in Section 3, we apply the proposed importance sampling method to the DRMs $\rho_{g_{\alpha,\gamma}}$ with distortion function $g_{\alpha,\gamma}(u) = \mathbb{1}_{[0,\alpha]}(u) \left(\frac{u}{\alpha}\right)^{1/\gamma} + \mathbb{1}_{(\alpha,1]}(u)$ and $\gamma \in \{1/2, 1, 2\}$. In terms of the DRMs, solvency capital is $\rho_{g_{\alpha,\gamma}}(E_0 - E_1)$. The underlying random factors are $R_B, Z, N, \xi_1, \xi_2, \dots$. We set $E_0 = 1000$ and $R_B = (V - 1/2)/10$, where V is beta distributed with parameters $(2, 2)$, i.e., $V \sim B(2, 2)$. The parameters of the stock are $\mu = 0.02$, $\sigma = 0.2$, $\Delta t = 1$, and we assume that half of the available capital is invested into the stock, i.e., $b = 0.5$. For the collective model, we assume that N is a Poisson random variable with parameter $\lambda = 5$, and $(\xi_k)_{k \geq 1}$ are independent exponentially distributed random variables with parameter $\vartheta' = 10$. The premium and reserve are 103% resp. 105% of the expected claims such that $\pi = 1.03\lambda\vartheta'$ and $v = 1.05\lambda\vartheta'$.

As in Section 3, the importance sampling estimates with the different ML approximations used in the measure changes are performed with $M = 2,000$ pivot samples for calibration of the approximation and determination of the importance sampling mixture distribution, and $N = 20,000$ samples of the mixture distribution for DRM estimation. As comparison a crude estimation with $M + N$ samples is implemented. We always use a discretization with $m = 20$. The ‘exact value’ of benchmarking is determined with a crude estimation with 1,000,000 samples and used to calculate the RMSE over $R = 1,000$ simulation runs.

4.3 Results

The results of the simulations are shown in Figure 5, which presents the ratio of the RMSEs of the crude method and the studied importance sampling methods. For further reference, the absolute RMSE of the estimates are provided in Figure 14 in Appendix A.7. For all DRMs, importance sampling with exact knowledge of the model leads to a significant reduction in

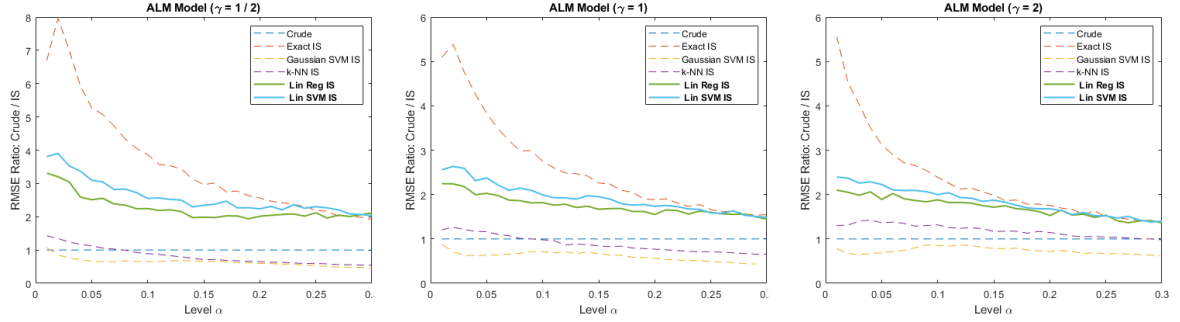


Figure 5: Ratio of the RMSE between the crude method and importance sampling method for the estimation of the considered DRMs for the evolution of the net asset value in the ALM model. The importance sampling method is implemented with the different approximation techniques considered in the paper.

the RMSE, especially for small α . The variance reduction decreases for larger α . For $\gamma = 1$, the exact method achieves the highest observed RMSE ratio of 8.8 for $\alpha = 0.01$. Importance sampling with linear regression leads to a maximum reduction of 5.8 for $\alpha = 0.01$. For the DRMs with concave ($\gamma = 2$) and convex ($\gamma = 1/2$) distortion functions, the linear SVM gives the best reduction in RMSE. For $\gamma = 2$, the best ratio obtained with full knowledge of the model is 5.55 for $\alpha = 0.01$, and with linear SVM, the maximum reduction is 2.39 for $\alpha = 0.01$. For $\gamma = 1/2$, the exact importance sampling method has the best ratio of 7.96 for $\alpha = 0.02$ and with the linear SVM approximation of 3.9 for $\alpha = 0.02$. Across all the different estimated DRMs, we see that the importance sampling methods with Gaussian SVM and k -NN regression can lead to a worse RMSE than the crude method. The worst ratio is observed in all cases with the Gaussian SVM with 0.51 as $\alpha = 0.28$ for $\gamma = 1$, 0.62 as $\alpha = 0.3$ for $\gamma = 2$ and 0.45 for $\gamma = 1/2$ with $\alpha = 0.3$.

In summary, the proposed method provides a good path to variance reduction. However, the ML approximation in the measure change needs to be chosen carefully, but k -fold validation seems to work quite well. The variance reduction becomes better the more the risk measure depends on extreme tail events. In the ALM case study, the most extreme parameter α was 0.01. We expect that the iterative procedure outlined in Section 3.3 would also lead to further improvements in variance reduction when the very extreme tail is considered.

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A Online Appendix

This is an online appendix that is provided as an electronic supplement to the paper.

A.1 Distortion Risk Measures

We review some facts related to risk measures and the special case of DRMs. Let \mathcal{X} denote the set of all bounded and measurable functions on the measurable space (Ω, \mathcal{F}) . Elements in \mathcal{X} model financial positions or insurance losses. We use the sign convention to interpret positive values as losses and negative values as gains. The axiomatic definition of risk measures goes back to Artzner et al. (1999); the notion of distortion risk measures (DRM) was developed by Wang (1996) and Acerbi (2002). DRMs are a subclass of comonotonic risk measures. The link of comonotonic risk measures and DRMs is briefly discussed in Appendix A.2. For an excellent overview on risk measures and DRMs we refer to Föllmer & Schied (2016).

A risk measure $\rho : \mathcal{X} \rightarrow \mathbb{R}$ is a functional that quantifies the risk of elements of \mathcal{X} :

Definition A.1. *A mapping $\rho : \mathcal{X} \rightarrow \mathbb{R}$ is called monetary risk measure if the following properties hold:*

(i) *Monotonicity:* If $X \leq Y$, $X, Y \in \mathcal{X}$, then $\rho(X) \leq \rho(Y)$.

(ii) *Cash-Invariance:* If $X \in \mathcal{X}$ and $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) + m$.

Risk measures may exhibit additional properties such as quasi-convexity, which in economic terms means that diversification of positions does not increase the measurements. This property can be shown to be equivalent to convexity:

Definition A.2. *A risk measure $\rho : \mathcal{X} \rightarrow \mathbb{R}$ is called convex, if for $X, Y \in \mathcal{X}$, $\lambda \in [0, 1]$*

$$\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y).$$

A DRM is defined as follows:

Definition A.3. (i) *A non decreasing function $g : [0, 1] \rightarrow [0, 1]$ with $g(0) = 0$ and $g(1) = 1$ is called distortion function.*

(ii) *Let \mathbf{P} be a probability measure on (Ω, \mathcal{F}) and g be a distortion function. The monetary*

risk measure $\rho_g : \mathcal{X} \rightarrow \mathbb{R}$ defined by

$$\rho_g(X) = \int_{-\infty}^0 [g(\mathbb{P}(\{X > x\})) - 1] dx + \int_0^{\infty} g(\mathbb{P}(\{X > x\})) dx$$

is called a DRM with respect to g .

If the distortion function g is concave we obtain a convex risk measure (see Föllmer & Schied (2016)):

Theorem A.4. *Consider the distortion function g and the corresponding DRM ρ_g . If g is concave, then the DRM ρ_g is a convex risk measure. If the underlying probability space is atomless, the converse implication is also true.*

DRMs can be expressed as mixture of quantiles. One must focus on the details of the continuity properties of the distortion function to obtain the correct representation, as shown in Dhaene et al. (2012).

Theorem A.5. (i) *Let g be a right continuous distortion function. Then the DRM $\rho_g(X)$ is given by*

$$\rho_g(X) = \int_{[0,1]} q_X^+(1-u) dg(u),$$

where $q_X^+(u) = \sup\{x | F_X(x) \leq u\}$.

(ii) *Let g be a left continuous distortion function. Then the DRM $\rho_g(X)$ is given by*

$$\rho_g(X) = \int_{[0,1]} q_X(1-u) dg(u) = \int_{[0,1]} q_X(u) d\bar{g}(u),$$

where $q_X(u) = \inf\{x | F(x) \geq u\}$ and $\bar{g}(u) = 1 - g(1-u)$, $u \in [0, 1]$.

Many important risk measures fall into the class of DRMs; for examples see Cherny & Madan (2008), Föllmer & Schied (2016), Weber (2018), and the Appendix A.3. Particularly important examples will be discussed here:

Example A.6. (i) *Let $g(u) = \mathbb{1}_{(\alpha,1]}(u)$, then ρ_g is the Value at Risk at level α , so that*

$$\rho_g(X) = V@R_\alpha(X) = \inf\{x | F(x) \geq 1 - \alpha\}.$$

(ii) *The distortion function $g(u) = \frac{u}{\alpha} \mathbb{1}_{[0,\alpha]}(u) + \mathbb{1}_{(\alpha,1]}(u)$ yields the Average Value at Risk at*

level α , i.e.,

$$\rho_g(X) = AV@R_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha V@R_\lambda(X) d\lambda.$$

(iii) The distortion function $g(u) = \left(\frac{u}{\alpha}\right)^\gamma \mathbb{1}_{[0,\alpha]}(u) + \mathbb{1}_{(\alpha,1)}(u)$ with $\alpha \in (0, 1]$ and $\gamma \in \mathbb{R}_{>0}$ generalizes the distortion function of the AV@R ($\gamma = 1$) and other special cases such as the hazard transform ($\gamma \geq 1, \alpha = 1$) and MAXV@R ($\gamma \in \mathbb{N}, \alpha = 1$).

If $\gamma \leq 1$, the distortion function $g_{\alpha,\gamma}$ is concave such that the corresponding DRM is convex. If $\gamma > 1$ the distortion function is convex. In this case, the resulting DRM is not convex.

Remark A.7. Every distortion function g can be decomposed in the convex combination of a left and right continuous distortion function (see Dhaene et al. (2012)), such that $g(u) = c_1 g_1(u) + c_2 g_2(u)$ with $c_1 + c_2 = 1$ and $c_1, c_2 \geq 0$. As a consequence, any distortion risk measure ρ_g with general distortion function can be expressed as convex combination $\rho_g(X) = c_1 \rho_{g_1}(X) + c_2 \rho_{g_2}(X)$. The decompositions of g and ρ_g is not unique, unless g is a step function. Bettels, Kim & Weber (2022) point out that a decomposition of g into a left and a right continuous step function and a continuous function is unique.

A.2 Comonotonic Risk Measures

We review the links between comonotonic risk measures, Choquet integrals and DRMs. More details can be found in Föllmer & Schied (2016). (Ω, \mathcal{F}) is a measurable space on which the financial positions in \mathcal{X} are defined.

Definition A.8. (i) Two measurable functions X, Y on (Ω, \mathcal{F}) are called comonotonic if

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0 \quad \forall (\omega, \omega') \in \Omega \times \Omega.$$

(ii) A risk measure $\rho : \mathcal{X} \rightarrow \mathbb{R}$ is called comonotonic if

$$\rho(X + Y) = \rho(X) + \rho(Y),$$

for comonotonic $X, Y \in \mathcal{X}$.

Comonotonic risk measures can be expressed as Choquet integrals with respect to capacities.

Definition A.9. (i) A map $c : \mathcal{F} \rightarrow [0, \infty)$ is called *monotone set function* if it satisfies the following properties:

a) $c(\emptyset) = 0$.

b) $A, B \in \mathcal{F}, A \subseteq B \Rightarrow c(A) \leq c(B)$.

If, in addition, $c(\Omega) = 1$, i.e., c is normalized, then c is called a *capacity*.

(ii) For $X \in \mathcal{X}$ the *Choquet integral* of X with respect to the monotone set function c is defined by

$$\int X dc = \int_{-\infty}^0 [c(\{X > x\}) - c(\Omega)] dx + \int_0^{\infty} c(\{X > x\}) dx.$$

The Choquet integral coincides with the Lebesgue integral if c is a σ -additive probability measure. The following characterization theorem can, for example, be found in Chapter 4 of Föllmer & Schied (2016).

Theorem A.10. A monetary risk measure $\rho : \mathcal{X} \rightarrow \mathbb{R}$ is comonotonic, if and only if there exists a capacity c on (Ω, \mathcal{F}) such that

$$\rho(X) = \int X dc.$$

DRMs are an important special case of the comonotonic risk measures. In this case, the capacity is defined in terms of a distorted probability measure P . The resulting capacity is absolutely continuous with respect to P , but typically not additive.

Definition A.11. (i) If P is a probability measure on (Ω, \mathcal{F}) and g is a distortion function, then

$$c^g(A) := g(P(A)), \quad A \in \mathcal{F},$$

is called a *distorted probability*.

(ii) A comonotonic risk measure $\rho(X) = \int X dc$ is called a *DRM*, if the capacity c can be expressed as a distorted probability.

A.3 Examples of DRMs

For further reference, we include a list of examples of distortion risk measure in Table 2 which was compiled in the online appendix of El Methni & Stupfler (2017).

Name	Distortion	Closed form	Reference
MINV@R	$1 - (1 - u)^n$	$-\mathbb{E}[\min\{-X_1, \dots, -X_n\}]$ $= \mathbb{E}[\max\{X_1, \dots, X_n\}]$	Cherny & Madan (2008) Föllmer & Schied (2016) Bannör & Scherer (2014)
MAXV@R	$u^{1/n}$	$-\mathbb{E}[Y_1]$ such that $\max\{Y_1, \dots, Y_n\} \sim -X$	Cherny & Madan (2008) Föllmer & Schied (2016) Bannör & Scherer (2014)
MINMAXV@R	$1 - (1 - u^{1/n})^n$	$-\mathbb{E}[\min\{Y_1, \dots, Y_n\}]$ such that $\max\{Y_1, \dots, Y_n\} \sim -X$	Cherny & Madan (2008) Föllmer & Schied (2016) Bannör & Scherer (2014)
MAXMINV@R	$(1 - (1 - u)^n)^{1/n}$	$-\mathbb{E}[Y_1]$ such that $\max\{Y_1, \dots, Y_n\}$ $\sim \min\{-X_1, \dots, -X_n\}$	Cherny & Madan (2008) Föllmer & Schied (2016) Bannör & Scherer (2014)
<i>RV@R</i> (Range <i>V@R</i>)	$\frac{u-\beta}{\alpha-\beta} \mathbb{1}_{\{\beta < u \leq \alpha\}} + \mathbb{1}_{\{u > \alpha\}}$ $0 < \beta < \alpha < 1$	$\frac{1}{\alpha-\beta} \int_{\beta}^{\alpha} V@R_{\lambda}(X) d\lambda$	Bignozzi & Tsanakas (2015) Weber (2018), Li et al. (2018)
Proportional hazard transform	$u^{1/\gamma}$ $\gamma > 1$	$\int_0^{\infty} (1 - F(x))^{1/\gamma} dx,$ if $X \geq 0$ a.s.	Wang (1995); Wang (1996) Guillen et al. (2018)
Dual power transform	$1 - (1 - u)^{\gamma}$ $\gamma > 1$	$\int_0^{\infty} 1 - F(x)^{\gamma} dx,$ if $X \geq 0$ a.s.	Wirch & Hardy (1999) Guillen et al. (2018)
Gini's principle	$(1 - \vartheta)u + \vartheta u^2$ $0 < \vartheta < 1$	$\mathbb{E}[X] + \frac{\vartheta}{2} \mathbb{E}[X - X_1]$	Yitzhaki (1982), Wozabal (2014) Guillen et al. (2018)
Exponential transform	$\frac{1 - \exp(-ru)}{1 - \exp(-r)}$ if $r > 0$ u if $r = 0$	-	El Methni & Stupfler (2017) Dowd, Cotter & Sorwar (2008)
Inverse S-shaped polynomial of degree 3	$a \left[\frac{u^3}{6} - \frac{\delta u^2}{2} + \left(\frac{\delta^2}{2} + \beta \right) u \right]$ $a = \left(\frac{1}{6} - \frac{\delta}{2} + \frac{\delta^2}{2} + \beta \right)^{-1}$ $0 < \delta < 1, \beta \in \mathbb{R}$	-	Guégan & Hassani (2014) El Methni & Stupfler (2017)
Beta family	$\int_0^u \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt$ $a, b > 0$	-	Samanthi & Sepanski (2018) Wirch & Hardy (1999)
Wang transform	$\Phi(\Phi^{-1}(u) - \Phi^{-1}(q))$ $0 < q < 1$	-	Wang (2000); Wang (2001) Wozabal (2014)

Table 2: Further examples of distortion risk measures of a random variable X . Table 1 of the online appendix of El Methni & Stupfler (2017) provides these examples of distortion functions; we include this table of examples as a convenient reference for the reader. In the third column, X_1, \dots, X_n denote independent copies of X , $n \in \mathbb{N}$; Y_1, \dots, Y_n are suitable iid random variables satisfying the conditions given in the third column of the table. B denotes the beta function, Φ, Φ^{-1} the distribution and quantile function of the standard normal distribution, respectively.

A.4 Asymptotics of Quantile Estimators in Importance Sampling

The importance sampling estimator in Section 2.1, eq. (2) is studied, along with other alternatives, in Glynn (1996). Obviously, we can equivalently rewrite the estimator in (2) as $\hat{q}_{F^*,N}(u) = \inf \left\{ x \in \mathbb{R} \mid \frac{1}{N} \sum_{i=1}^N \frac{dF}{dF^*}(X_i) \mathbf{1}_{\{h(X_i) \leq x\}} \geq u \right\}$. Setting $F^* = F$, the estimator coincides with the crude Monte Carlo estimator of quantiles, the empirical quantile. We want to analyze under which conditions the estimator in eq. (2) is finite.

For this purpose, we first analyse a deterministic problem. Let $\alpha_1, \dots, \alpha_N \in \mathbb{R}$, $\gamma_i \geq 0$ for $i = 1, 2, \dots, N$ and $z > 0$. Then

$$\inf \left\{ x \in \mathbb{R} \mid \sum_{\alpha_i > x} \gamma_i \leq z \right\} \in \mathbb{R} \iff \sum \gamma_i > z.$$

To see this, we denote the inf-term on the left hand side by q and observe that $q = -\infty$ is equivalent to $\sum_{\alpha_i > x} \gamma_i \leq z$ for all x which simply means that $\sum \gamma_i \leq z$. This would already prove the claim, if $q = \infty$ was impossible. But the latter equality is equivalent to $\sum_{\alpha_i > x} \gamma_i > z$ for all x ; for large enough x , the sum is empty and equal to 0, contradicting $z > 0$.

The simple characterization implies for $u \in (0, 1)$ that

$$\hat{q}_{F^*,N}(u) \in \mathbb{R} \iff \frac{1}{N} \sum_{i=1}^N \frac{dF}{dF^*}(X_i) > 1 - u \quad (7)$$

If we assume that $(X_1, h(X_1)), \dots, (X_N, h(X_N))$ are independent, identically distributed samples from F^* , then by a law of large numbers $\frac{1}{N} \sum_{i=1}^N \frac{dF}{dF^*}(X_i) \rightarrow \mathbf{E}_{F^*} \left[\frac{dF}{dF^*}(X) \right] = 1$, thus eq. (7) is satisfied for N large enough.

The asymptotic normality of the estimator $\hat{q}_{F^*,N}(\alpha)$ can be shown, if the following assumptions holds. This is stated in Theorem 2.1 of Ahn & Shyamalkumar (2011), generalizing Glynn (1996).

Assumption A.12. Let G_Y, G^* be the distribution functions of $h(X)$ under F, F^* , such that $G_Y = F \circ h^{-1}, G^* = F^* \circ h^{-1}$.

Assume that for $u \in (0, 1)$ the following properties hold:

(A1) G_Y is absolutely continuous with respect to G^* .

(A2) G^* is continuous at $q_Y(u)$.

(A3) G_Y has a strictly positive first derivative at $q_Y(u)$.

(A4) $\frac{dG}{dG^*}(\cdot)$ is a function of finite variation on compacts and has finite negative variation on (y, ∞) for all $y \in \mathbb{R}$.

(A5) $\frac{dG}{dG^*}(\cdot)$ is right continuous.

(A6) There exists a $\lambda \in (0, 1/2]$ such that

$$\int_y^\infty (1 - G^*(x-))^{1/2-\lambda} d \left| \frac{dG}{dG^*}(x) \right| < \infty \quad \forall y \in \mathbb{R}.$$

Remark A.13. These assumptions of Ahn & Shyamalkumar (2011) are weaker than the assumptions of Glynn (1996) to obtain the implications in Theorem 2.1. Glynn (1996) assumes that (A1) to (A3) hold and $\mathbb{E}_{F^*} \left[\left| \frac{dF}{dF^*}(X) \right|^3 \right] < \infty$. The latter is replaced by assumption (A6), together with the technical conditions (A4) and (A5); here, $|\cdot|$ denotes total variation.¹ Ahn & Shyamalkumar (2011) show that if $\mathbb{E}_{F^*} \left[\left| \frac{dG}{dG^*} \right|^{2+\delta} \right] < \infty$ holds for some $\delta > 0$, then (A6) is satisfied for $\lambda \in (0, \delta/(4 + 2\delta))$.

Proposition A.14. If Assumption A.12 holds, we obtain for $u \in (0, 1)$:

(i) $\mathbb{E}_{F^*} \left[\frac{dF}{dF^*}(X)^2 \mathbb{1}_{\{h(X) \in (q_Y(u), \infty)\}} \right] \geq (1 - u)^2$

(ii) Suppose that F and F^* are equivalent. Then

$$\mathbb{E}_{F^*} \left[\frac{dF}{dF^*}(X)^2 \mathbb{1}_{\{h(X) \in (q_Y(u), \infty)\}} \right] \geq (1 - G^*(q_Y(u)))^{-1}$$

Proof. (i) By assumption $G_Y = F \circ h^{-1}$ is continuous at $q_Y(u)$, such that $G(q_Y(u)) = u$. The inequality is thus a simple consequence of Jensen's inequality.

(ii) The function $f(x) = 1/x$ is convex function on $\mathbb{R}_{>0}$. Hence, by Jensen's inequality we have

$$\mathbb{E}_{F^*} \left[\frac{dF}{dF^*}(X)^2 \mathbb{1}_{\{h(X) > q_Y(u)\}} \right] = \mathbb{E} \left[\left(\frac{dF^*}{dF}(X) \right)^{-1} \mathbb{1}_{\{h(X) > q_Y(u)\}} \right] \geq (1 - G^*(q_Y(u)))^{-1}.$$

□

¹If $f : \mathbb{R} \mapsto \mathbb{R}$ is a function of bounded variation, there exist two increasing functions $f^+, f^- : \mathbb{R} \mapsto \mathbb{R}$ with $f = f^+ - f^-$, and $|f| := f^+ + f^-$. The former is called the Jordan decomposition of f and is closely related to the Hahn decomposition of signed measures. For details we refer to Folland (1999).

We now consider the estimation of the quantile $q_Y(1 - \alpha)$, $\alpha \in (0, 1)$, by $\hat{q}_{F_\vartheta, N}(1 - \alpha)$, the estimator defined in Section 2.1, eq. (2). According to Theorem 2.1 we should choose F_ϑ such that

$$\mathbf{E}_{F_\vartheta} \left[\frac{dF}{dF_\vartheta}(X)^2 \mathbf{1}_{\{h(X) > q_Y(1-\alpha)\}} \right] \quad (8)$$

is small. We consider exponential twists given in eq. (3). The following standard result for the cumulant generating function is useful:

Lemma A.15. *Letting $F : \mathbb{R}^d \rightarrow [0, 1]$ be the distribution function of X , $h : \mathbb{R}^d \rightarrow \mathbb{R}$ a measurable function. If $\psi(\vartheta + t) = \log(\mathbf{E}[\exp((\vartheta + t)h(X))]) < \infty$ for all t in some neighborhood of 0, then $\psi'(\vartheta) = \mathbf{E}_{F_\vartheta}[h(X)]$, where F_ϑ is the family of distributions defined in (3).*

To find an appropriate parameter ϑ to make (8) small, we take an approach like in Sun & Hong (2009). By the definition of F_ϑ samples in the right tail are more likely to occur when $\vartheta > 0$, which also indicates that to minimize (8) we should choose $\vartheta > 0$. We observe that

$$\begin{aligned} \mathbf{E}_{F_\vartheta} \left[\frac{dF}{dF_\vartheta}(X)^2 \mathbf{1}_{\{h(X) > q_Y(1-\alpha)\}} \right] &= \mathbf{E} \left[\frac{dF}{dF_\vartheta}(X) \mathbf{1}_{\{h(X) > q_Y(1-\alpha)\}} \right] = \mathbf{E} [\exp(\psi(\vartheta) - \vartheta h(X)) \mathbf{1}_{\{h(X) > q_Y(1-\alpha)\}}] \\ &\leq \exp(\psi(\vartheta) - \vartheta q_Y(1 - \alpha)) \cdot \mathbf{P}(h(X) > q_Y(1 - \alpha)). \end{aligned}$$

Minimizing the upper bound is then equivalent to minimizing $\psi(\vartheta) - \vartheta q_Y(1 - \alpha)$, from which we obtain a first order condition using Lemma A.15

$$q_Y(1 - \alpha) = \mathbf{E}_{F_\vartheta} [h(X)]. \quad (9)$$

In their paper, Sun & Hong (2009) show that this approach yields a strict reduction of the objective function (8):

Theorem A.16. *Consider the situation as described above. Assume there exists $\varepsilon > 0$ such that G_Y is differentiable with strictly positive derivative on $(q_Y(1 - \alpha) - \varepsilon, q_Y(1 - \alpha) + \varepsilon)$. Further suppose that $q_Y(1 - \alpha) > \mathbf{E}[h(X)]$, $\frac{dF}{dF_{\vartheta^*}}(x) = \exp(\psi(\vartheta^*) - \vartheta^* h(x))$, and ϑ^* be chosen such that $q_Y(1 - \alpha) = \mathbf{E}_{F_{\vartheta^*}}[h(X)]$. Then $\mathbf{E}_{F_{\vartheta^*}} \left[\frac{dF}{dF_{\vartheta^*}}(X)^2 \mathbf{1}_{\{h(X) > q_Y(1-\alpha)\}} \right] < \mathbf{E} [\mathbf{1}_{\{h(X) > q_Y(1-\alpha)\}}]$.*

A.5 Tools from Machine Learning

For convenience, we briefly review the considered ML regression techniques and the methodology of k -fold validation. A good introduction to machine learning is Shalev-Shwartz & Ben-

David (2014)). In our simulation algorithm, pivot samples $S = (X_1, h(X_1)), \dots, (X_M, h(X_M)) = (X_i, h(X_i))_{i=1, \dots, M}$ are used as training data.

A.5.1 Linear Predictors

We briefly review linear prediction; this is based on Section 9.2 of Shalev-Shwartz & Ben-David (2014). For regressions, we consider the hypothesis class

$$\mathcal{H}_{lin} = \left\{ x \mapsto \langle w, x \rangle + b \mid w \in \mathbb{R}^d, b \in \mathbb{R} \right\} \quad (10)$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. One approach is to determine w^*, b^* by empirical risk minimization (ERM) with quadratic loss function. The empirical risk is thus given by $L_S(h) = \sum_{i=1}^M (\langle w, X_i \rangle + b - h(X_i))^2$, where $h \in \mathcal{H}_{lin}$ is the predictor corresponding to w, b . Optimal w^*, b^* are determined by $w^*, b^* = \arg \min_{w, b} \sum_{i=1}^M (\langle w, X_i \rangle + b - h(X_i))^2$. As is well known, the first order conditions leads to linear problem. Shalev-Shwartz & Ben-David (2014) discuss the application of linear programming and perceptrons (cf. Rosenblatt (1958)).

A.5.2 Polynomial Predictors

Again, we refer for more details, Shalev-Shwartz & Ben-David (2014), Section 9.2.2. To illustrate the main idea, we assume that the dimension of the training patterns is $d = 1$. The hypothesis class of polynomial predictors with degree k is given as

$$\mathcal{H}_{poly}^k = \left\{ x \mapsto p(x) \mid w \in \mathbb{R}^{k+1} \right\}$$

where $p(x) = w_0 + w_1x + w_2x^2 + \dots + w_kx^k$. Obviously, polynomial predictors can be seen as the application of linear hypotheses to features which are obtained as a transformations of the original input patterns, in this case leading to monomials as features. Namely, setting $\psi(x) = (1, x, x^2, \dots, x^k)$, we have $p(x) = \langle w, \psi(x) \rangle$. We can thus apply the same methods on the transformed sample $S' = (\psi(X_i), h(X_i))_{i=1, \dots, M}$ as in the case of linear predictors with ERM specified by $w^* = \arg \min_w \sum_{i=1}^M (\langle w_i, \psi(X_i) \rangle + h(X_i))^2$.

A.5.3 Support Vector Machines

Support vector machines can be used for classification and regression purposes. A good overview on classification is Chapter 15 of Shalev-Shwartz & Ben-David (2014). An early extension to

regression tasks is Drucker et al. (1996). For more details see and Chapter 6 in Vapnik (1999), Chapter 2 in Huang, Kecman & Kopriva (2006), or the tutorial article Smola & Schölkopf (2004) which form the basis for our brief review.

Linear Support Vector Machine Regression First, we consider again the linear predictor hypothesis class (10). A support vector machine regression considers the optimization problem

$$(w^*, b^*) = \arg \min_{w, b} \frac{1}{2} \|w\|^2$$

$$\text{subject to } |h(X_i) - \langle w, X_i \rangle - b| \leq \varepsilon,$$

where $\varepsilon > 0$ is a parameter controlling the tolerated distance of the samples to the predictor; within the tolerance bound, the flatness of the solution is minimized. As the solution to the optimization problem above may not exist, the soft margin concept introduces the slack variables $\xi, \bar{\xi} \in \mathbb{R}^M$ and considers instead

$$(w^*, b^*, \xi^*, \bar{\xi}^*) = \arg \min_{w, b, \xi, \bar{\xi}} \frac{1}{2} \|w\|^2 + C \sum_{i=1}^M (\xi_i + \bar{\xi}_i),$$

$$\text{subject to } h(X_i) - (\langle w, X_i \rangle - b) \leq \varepsilon + \xi_i,$$

$$(\langle w, X_i \rangle + b) - h(X_i) \leq \varepsilon + \bar{\xi}_i,$$

$$\xi_i, \bar{\xi}_i \geq 0.$$

To approximate the solution of the soft margin optimization we use in this paper a sequential optimization described in Platt (1998) and Fan et al. (2005).

The Kernel Trick The summary in this section is based on chapter 16 of Shalev-Shwartz & Ben-David (2014), Section 6.3 of Vapnik (1999), Section 2.2 of Huang, Kecman & Kopriva (2006), and Smola & Schölkopf (2004). When generalizing support vector machines (SV) to nonlinear predictors, the same approach as outlined for polynomial predictors can be taken. Instead of considering linear hypotheses on the input space, one considers instead the concatenation of a unknown linear function and a known mapping from the input space to a feature space. Machine learning then determines a suitable linear predictor on the feature space. Good feature space can be very high-dimensional and the algorithm might become infeasible.

In the case of support vector machines, a computationally cheaper way is available which

relies on the following observation. If linear predictors are learnt on an Euclidian space using SV optimization, the solution can be determined if scalar products of all elements of the domain of the linear predictors can be computed. Consider, for example, the input space \mathbb{R} and the transformation to features $\psi : \mathbb{R} \mapsto \mathbb{R}^m$. Replacing the original training samples $S = (X_i, h(X_i))_{i=1, \dots, M}$ by $\hat{S} = (\psi(X_i), h(X_i))_{i=1, \dots, M}$, we seek a SV linear predictor computed from \hat{S} . Since the solution can be computed from the knowledge of scalar products of features $\langle \psi(x), \psi(x') \rangle = K(x, x')$ which are labeled by inputs, it suffices to specify the corresponding kernel $K : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, but explicit knowledge of ψ is not required. In this article, we again use the sequential optimization described, e.g., in Fan et al. (2005) with two commonly used kernel functions:

Example A.17. (i) *Polynomial kernel:* The kernel $K(x, x') = (1 + \langle x, x' \rangle)^k$ corresponds to

$$K(x, x') = \langle \psi(x), \psi(x') \rangle = \sum_{J \in \{0,1\}^k} \prod_{i=1}^k x_{J_i} \prod_{i=1}^k x'_{J_i},$$

where we define $x_0 = x'_0 = 1$. Then $\psi(x)$ has as components monomials up to degree k , and the SV machine will learn a polynomial predictor.

(ii) *Gaussian kernels:* The kernel

$$K(x, x') = \exp\left(-\frac{\|x - x'\|^2}{2\sigma}\right),$$

for $\sigma > 0$, is called *Gaussian kernel*. The Gaussian kernel corresponds to the embedding $\psi(x)$ with the components

$$\psi(x)_i = \frac{1}{\sqrt{i!}} \exp\left(\frac{x^2}{2}\right) x^i.$$

A.5.4 k -fold Cross Validation

Based on Section 11.2 of Shalev-Shwartz & Ben-David (2014), we briefly describe k -fold cross validation. The training of different methods was already discussed, now we need a strategy to select among these methods. For this purpose the training data S is split in sets S_1, \dots, S_k of size M/k (where k divides M which can easily be realized in the implementation) such that $S_j := (X_i, h(X_i))_{i=j \cdot \frac{M}{k} + 1, \dots, (j+1) \cdot \frac{M}{k}}$. Assume that $r \in \{1, \dots, R\}$ enumerates the different methods considered and/or parameters of these methods, and let $A_r(S)$ be the output of the algorithm trained on the training data S resulting in the predictor h_r . For each r the algorithm

Hypothesis Class	Hyperparameter	Stop Criterion
Linear Predictors	-	-
Polynomial Predictors of degree q_1	ordered increasing in $q_1 \in \{2, 3, \dots\}$	Overfitting observed
Linear SVM	-	-
Polynomial SVM of degree q_2	ordered increasing in $q_2 \in \{2, 3, \dots\}$	Overfitting observed Fitting computational unfeasible
Gaussian SVM	-	-
k -NN Regression	ordered increasing in $k \in \{1, 2, 3, \dots\}$	Overfitting observed

Table 3: Overview of the hypothesis classes and order of the hypothesis classes considered in the k -fold validation. The abortion criterion determines the largest hyperparameter considered for the hypothesis classes.

can alternatively be trained on the training data $S \setminus S_j$, $j \in \{1, \dots, k\}$ with output hypothesis $h_{r,j}$. The individual predictors $h_{r,j}$ are validated on the remaining fold of training data, i.e.,

$$\text{error}(r) = \frac{1}{k} \sum_{i=1}^k L_{S_i}(h_{r,i}) = \frac{1}{k} \sum_{i=1}^k \sum_{x \in S_i} l(h(x), h_{r,i}(x)),$$

where $l(\cdot, \cdot)$ is the considered loss function. In our implementation we will use $l(x, y) = (x - y)^2$ for the purpose of error measurement, although this loss function is not used in SVMs or NN. From the estimated errors of the predictors h_r we can then choose the one which is performing best. The hypothesis classes from Sections 3 & 4 are displayed in Table 3.

A.6 Proofs and Calculations

A.6.1 Appendix to Section 2.3.1

Auxiliary computations. Suppose that Assumption A.12 holds. For large enough N_i we use the approximation from Theorem 2.1 for all i , i.e.,

$$\hat{q}_{F_i, N_i}(1 - \alpha_i) \sim \mathcal{N} \left(q_Y(1 - \alpha_i), \frac{\mathbb{E}_{F_i} \left[\frac{dF}{dF_i}(X)^2 \mathbf{1}_{\{h(X) > q_Y(1 - \alpha_i)\}} \right] - \alpha_i^2}{NG'_Y(q_Y(1 - \alpha_i))^2} \right).$$

For $u \in [\alpha_i, \alpha_{i+1})$ we have

$$\begin{aligned} \mathbb{E} [(q_Y(1 - u) - \hat{q}_Y(1 - u))^2] &= \mathbb{E} [(q_Y(1 - u) - \hat{q}_{F_i, N_i}(1 - \alpha_i))^2] \\ &\approx \mathbb{E} \left[\left(q_Y(1 - u) - q_Y(1 - \alpha_i) - \sqrt{\frac{\mathbb{E}_{F_i} \left[\frac{dF}{dF_i}(X)^2 \mathbf{1}_{\{h(X) > q_Y(1 - \alpha_i)\}} \right] - \alpha_i^2}{N_i G'_Y(q_Y(1 - \alpha_i))^2}} Z_i \right)^2 \right] \end{aligned}$$

$$\begin{aligned}
&= (q_Y(1-u) - q_Y(1-\alpha_i))^2 - 2(q_Y(1-u) - q_Y(1-\alpha_i)) \sqrt{\frac{\mathbb{E}_{F_i} \left[\frac{dF}{dF_i}(X)^2 \mathbb{1}_{\{h(X) > q_Y(1-\alpha_i)\}} \right] - \alpha_i^2}{NG'_Y(q_Y(1-\alpha_i))^2}} \mathbb{E}[Z_i] \\
&+ \frac{\mathbb{E}_{F_i} \left[\frac{dF}{dF_i}(X)^2 \mathbb{1}_{\{h(X) > q_Y(1-\alpha_i)\}} \right] - \alpha_i^2}{NG'_Y(q_Y(1-\alpha_i))^2} \mathbb{E}[Z_i^2] \\
&= (q_Y(1-u) - q_Y(1-\alpha_i))^2 + \frac{\mathbb{E}_{F_i} \left[\frac{dF}{dF_i}(X)^2 \mathbb{1}_{\{h(X) > q_Y(1-\alpha_i)\}} \right] - \alpha_i^2}{NG'_Y(q_Y(1-\alpha_i))^2}
\end{aligned}$$

where $Z_i, i \in \{0, 1, \dots, m\}$ are iid standard normals. With this we obtain

$$\begin{aligned}
&\int_0^1 \mathbb{E}[(q_Y(1-u) - \hat{q}_Y(1-u))^2] dg(u) = \sum_{i=0}^m \int_{\alpha_i}^{\alpha_{i+1}} \mathbb{E}[(q_Y(1-u) - \hat{q}_{F_i, N_i}(1-\alpha_i))^2] dg(u) \\
&\approx \sum_{i=0}^m \int_{\alpha_i}^{\alpha_{i+1}} (q_Y(1-u) - q_Y(1-\alpha_i))^2 dg(u) + \frac{\mathbb{E}_{F_i} \left[\frac{dF}{dF_i}(X)^2 \mathbb{1}_{\{h(X) > q_Y(1-\alpha_i)\}} \right] - \alpha_i^2}{N_i G'_Y(q_Y(1-\alpha_i))^2} (g(\alpha_{i+1}-) - g(\alpha_i)).
\end{aligned}$$

□

Proof of Equation (6). Let

$$c_i := \frac{\mathbb{E}_{F_i} \left[\frac{dF}{dF_i}(X)^2 \mathbb{1}_{\{h(X) > q_Y(1-\alpha_i)\}} \right] - \alpha_i^2}{G'_Y(q_Y(1-\alpha_i))} (g(\alpha_{i+1}-) - g(\alpha_i)).$$

The optimization problem becomes to minimize $\sum_{i=0}^m \frac{c_i}{N_i}$ under the constraint $(\sum_{i=0}^m N_i) - N = 0$ with $c_i \geq 0$, since g is increasing and according to Proposition A.14. The Lagrangian of the optimization problem is $\mathcal{L}(N_0, N_1, \dots, N_m; \lambda) = \sum_{i=0}^m \frac{c_i}{N_i} + \lambda (\sum_{i=0}^m N_i - N)$ with gradient

$$\nabla \mathcal{L}(N_0, N_1, \dots, N_m; \lambda) = \left(-\frac{c_0}{N_0^2} + \lambda \quad -\frac{c_1}{N_1^2} + \lambda \quad \dots \quad -\frac{c_m}{N_m^2} + \lambda \quad \sum_{i=0}^m N_i - N \right)^T \stackrel{!}{=} 0.$$

We rewrite the first $m+1$ equations as $\sqrt{\frac{c_i}{\lambda}} = N_i, i = 0, 1, \dots, m$, and plug this into the last equation to obtain $\sum_{i=0}^m \sqrt{\frac{c_i}{\lambda}} = N$ which is equivalent to $\lambda = \left(\frac{1}{N} \sum_{i=0}^m \sqrt{c_i}\right)^2$. This yields the critical point $N_i = N \frac{\sqrt{c_i}}{\sum_{i=0}^m \sqrt{c_i}}, i = 0, 1, \dots, m$. To show that this is the minimum under the constraint it suffices to verify that $\mathcal{L}(N_0, N_1, \dots, N_m; \lambda)$ is a convex function in (N_0, N_1, \dots, N_m) . Rewriting $\mathcal{L}(N_0, N_1, \dots, N_m; \lambda) = \sum_{i=0}^m \frac{c_i}{N_i} + \lambda N_i - \frac{N}{m+1}$, we observe that the functions $\mathcal{L}_i(N') = \frac{c_i}{N'} + \lambda N' - \frac{N}{m+1}$ are each the sum of two convex functions and therefore itself convex if $N' \in \mathbb{R}_+$. It follows that $\mathcal{L}(N_0, N_1, \dots, N_m; \lambda)$ is a convex function, implying that the critical point is a minimum. □

A.7 Additional Plots

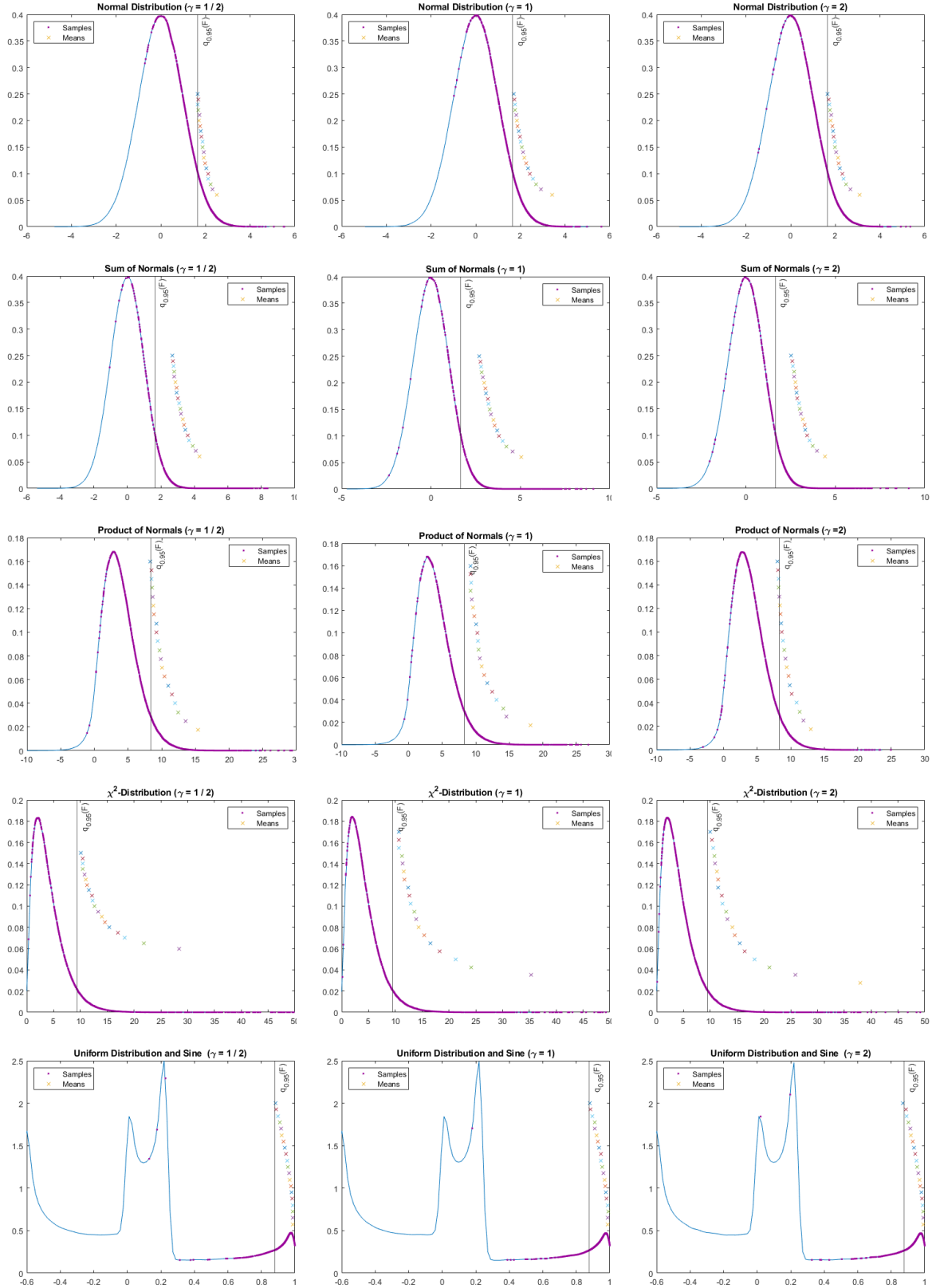


Figure 6: 200 samples drawn from the mixture distribution plotted on the underlying distribution of the model Y for the case studies (1) to (6). To approximate the mixture weights and optimal mixture components $M = 20,000$ pivot samples were drawn. For the estimation of the quantile and DRM $N = 100,000$ samples are drawn from the mixture distribution. For further details see Section 3.2.1.

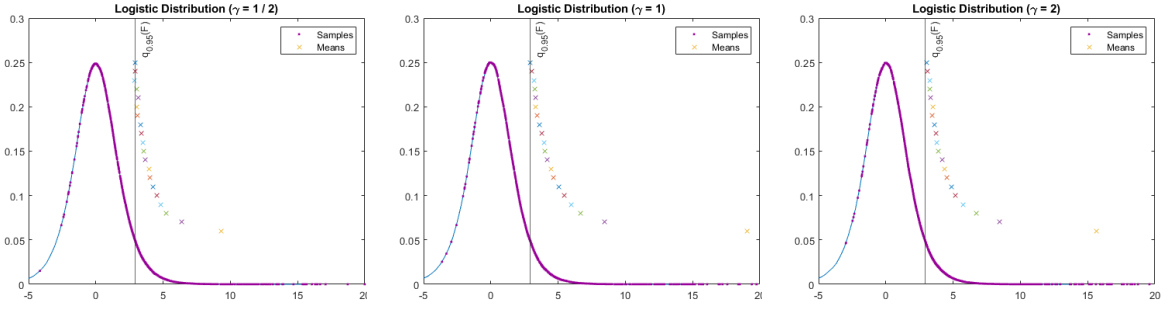


Figure 7: Continuation of Figure 2.

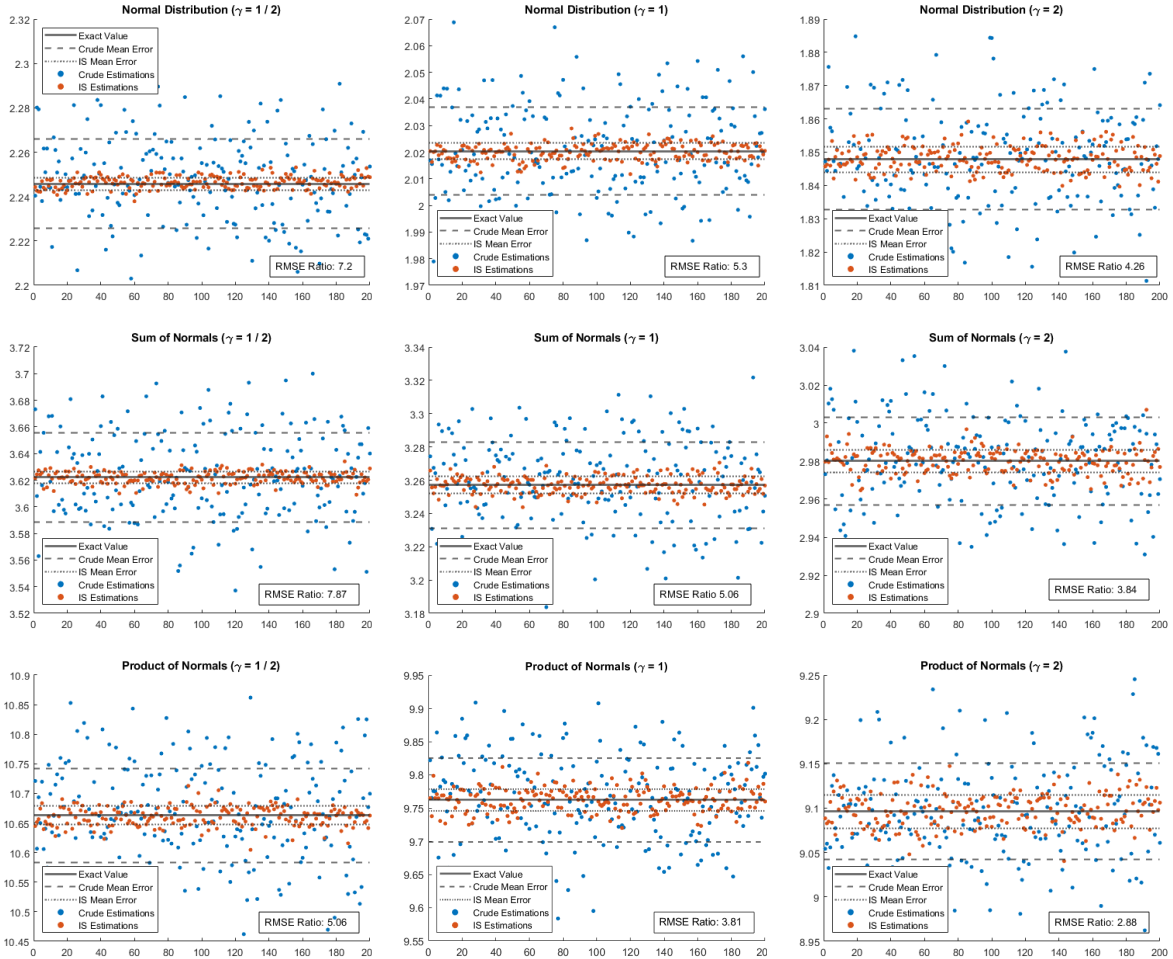


Figure 8: 200 estimations with a crude Monte Carlo estimation and the proposed importance sampling method for the models (1) to (6) and the considered DRMs $\rho_{g_{\gamma,\alpha}}$, $\gamma \in \{1/2, 1, 2\}$, $\alpha = 0.05$. Also shown is the “exact value”, which is calculated with a crude Monte Carlo estimation over 10,000,000 samples, the estimated root mean square error of the estimation around the exact value and the ratio of the root mean square error of the crude method and importance sampling method.

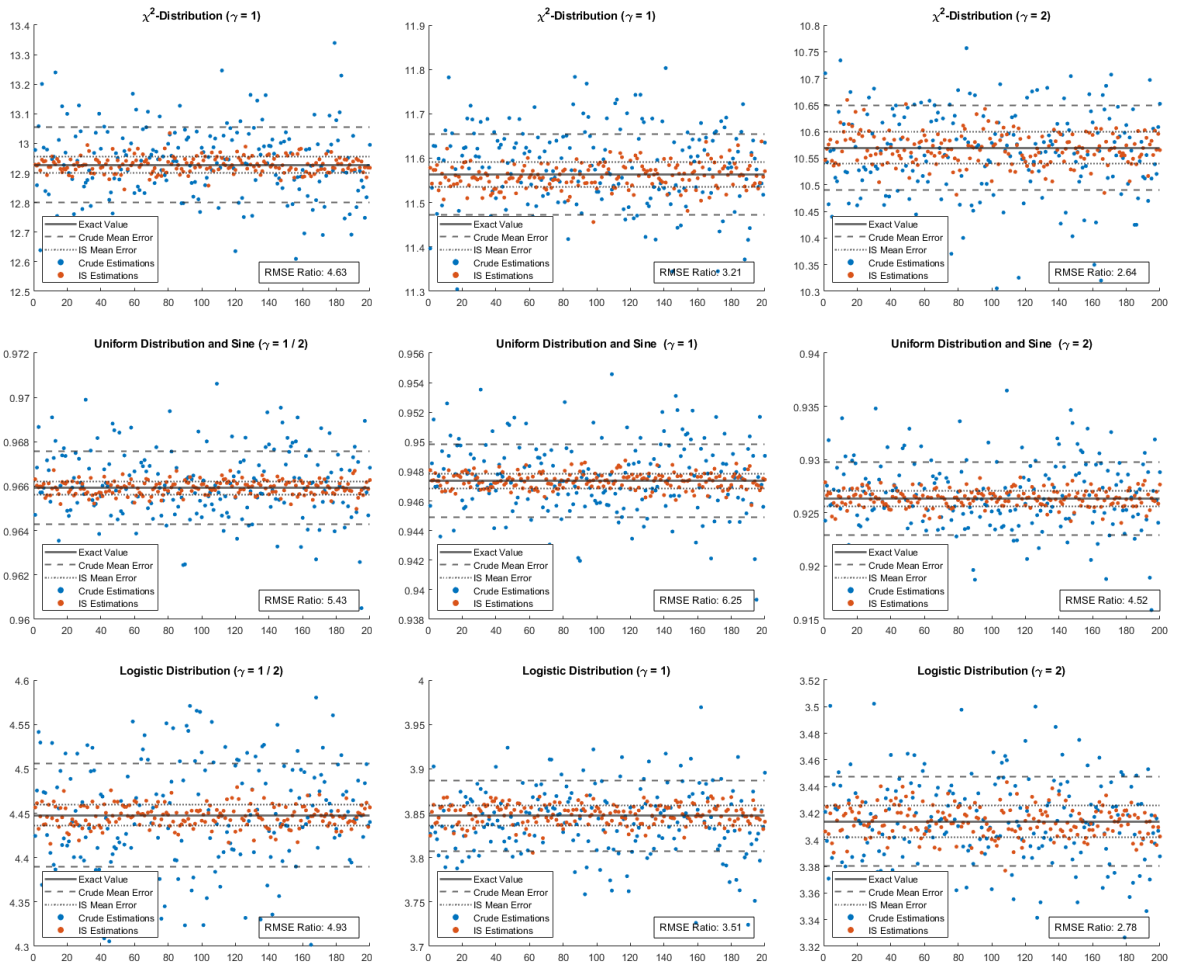


Figure 9: Continuation of Figure 8.

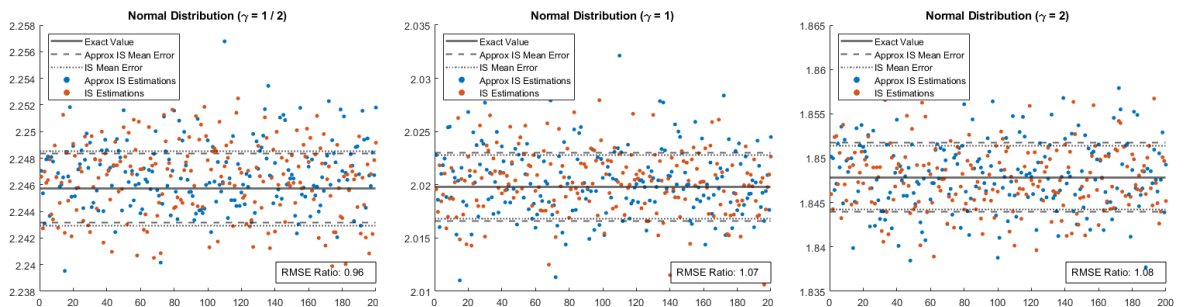


Figure 10: 200 estimations with the importance sampling method with exact knowledge of the model and the importance sampling method with a by k -fold validation chosen approximation of the model for the case studies (1) to (6) and the considered DRMs $\rho_{g,\alpha}$, $\gamma \in \{1/2, 1, 2\}$, $\alpha = 0.05$. Also shown is the “exact value”, which is calculated with a crude Monte Carlo estimation over 10,000,000 samples, the estimated root mean square error of the estimation around the exact value and the ratio of the root mean square error between both importance sampling methods.

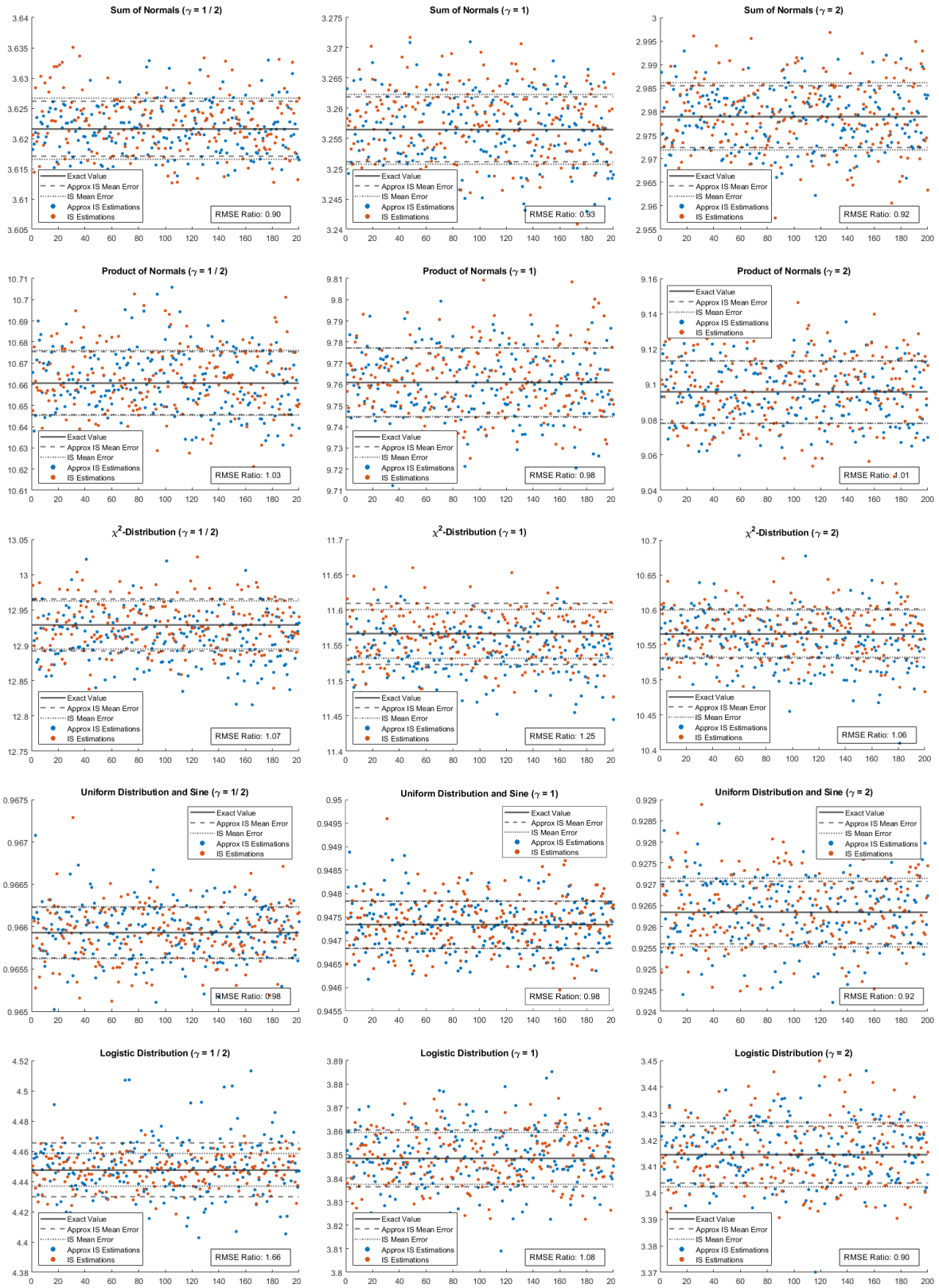


Figure 11: Continuation of Figure 10.

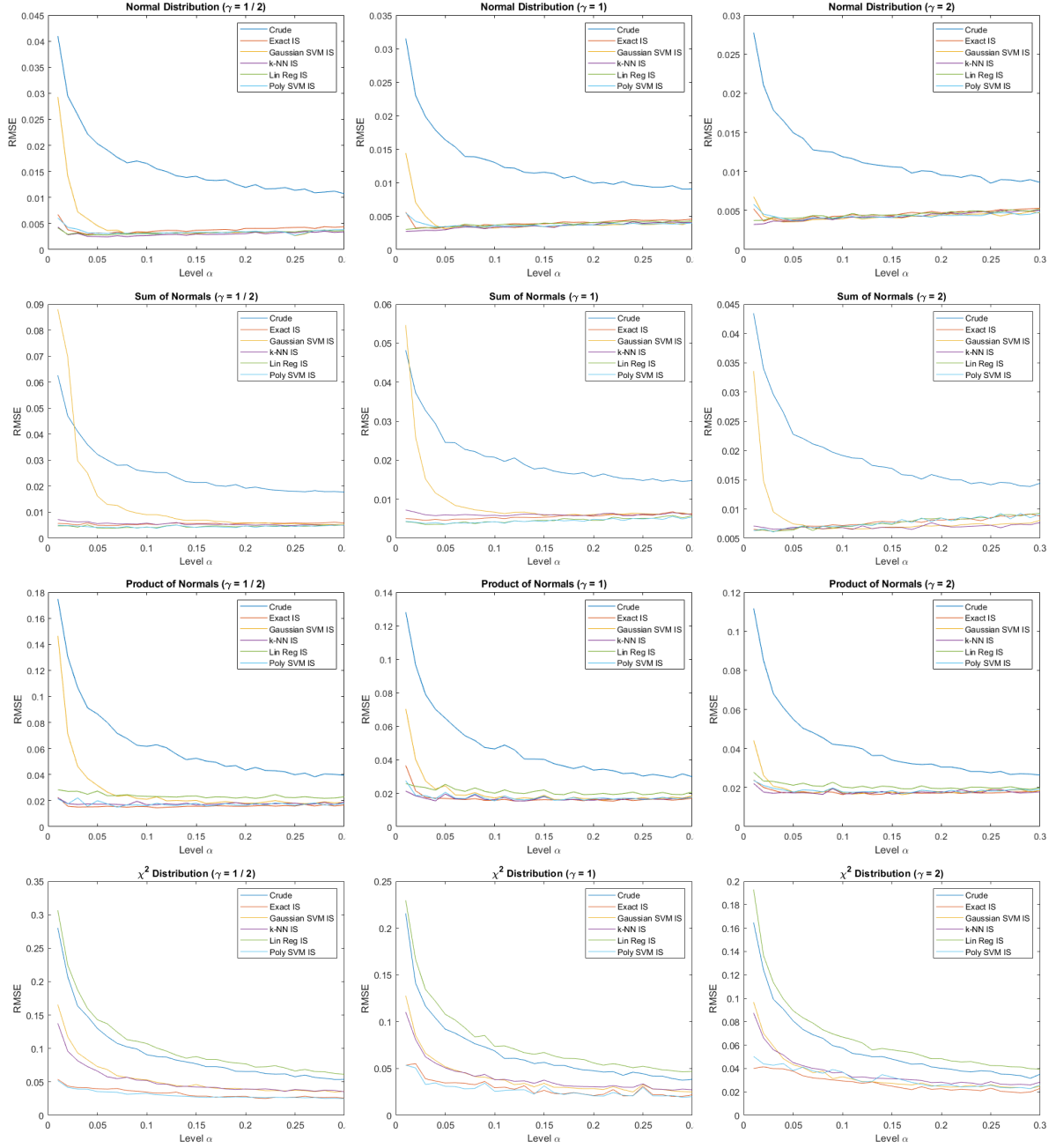


Figure 12: RMSE for the estimation of the DRMs $\rho_{g_{\gamma,\alpha}}$, with $\gamma \in \{1/2, 1, 2\}$, $\alpha \in [0.01, 0.3]$, for the models (1) to (6). The DRMs are estimated with a crude Monte Carlo method and the proposed importance sampling method with the different approximations of the black box models used in the paper.

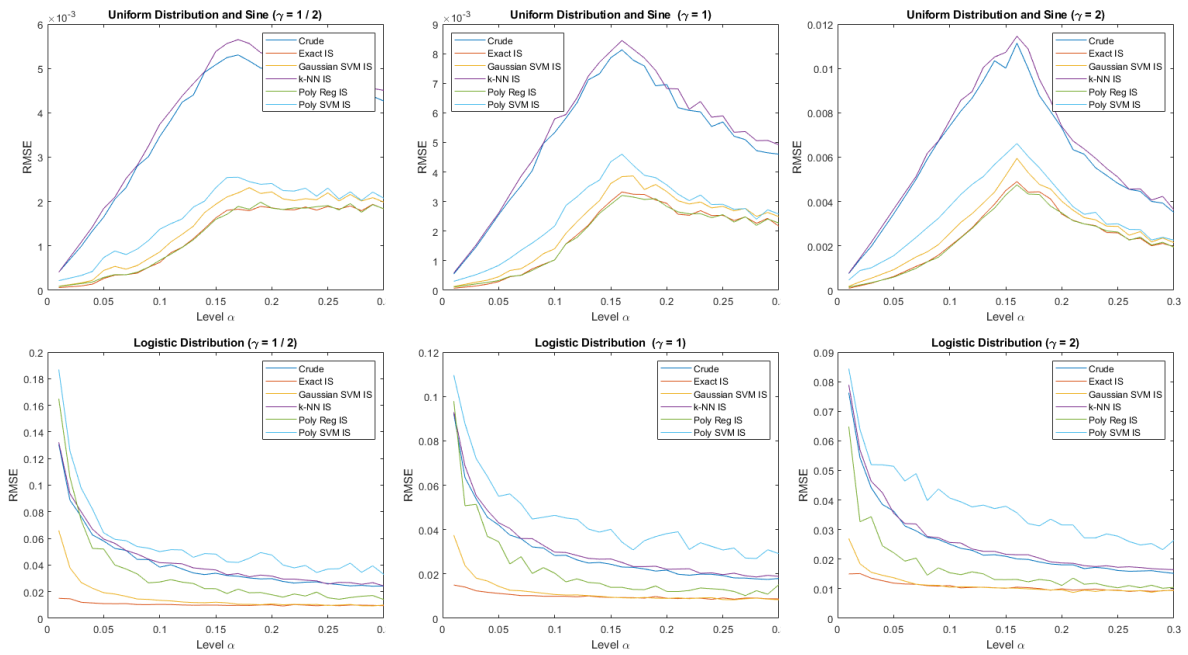


Figure 13: Continuation of Figure 12.

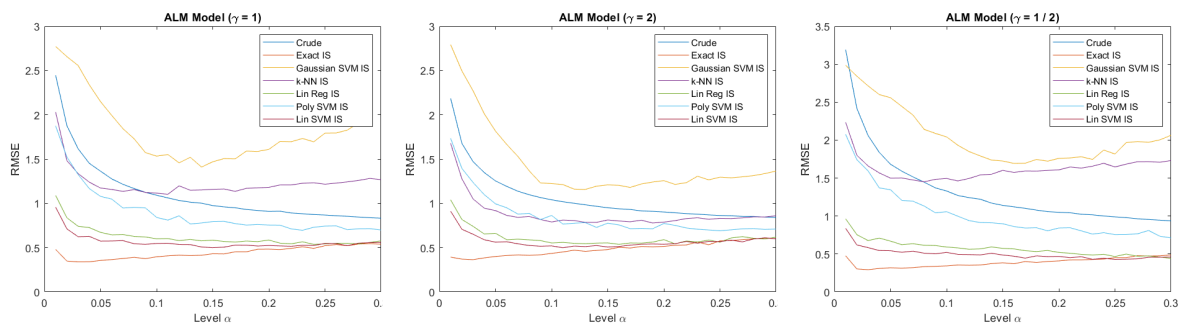


Figure 14: RMSE of the crude method and the different importance sampling methods of the considered DRMs for the evolution of the net asset value in the ALM model. The importance sampling methods are implemented with the different approximation techniques considered in the paper.